Incomplete Asset Market View of the Exchange Rate Determination *

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Abstract

We completely characterize the fundamental relationship between the exchange rate and the asset pricing in the two denomination currencies involved when markets are incomplete. Assuming arbitrage-free, perfectly integrated, frictionless but potentially incomplete financial markets, the exchange rate is equal to the ratio of countries' minimum-variance stochastic discount factors *if and only if* every exchange rate risk can be separately contracted in asset markets, i.e., exchange rate risks are completely disentangled. Abstracting from structural assumptions, the entanglement of exchange rate risks presents a novel and pure market-based rationale for a disconnection between prices and quantities in the international economy. Our study demonstrates when and how the influential asset market view of the exchange rate does not pose strong implications from the exchange rate dynamics on the macroeconomic fundamentals and their pricing.

JEL-Classification: F31, G15, G10.

Keywords: Exchange Rates, Incomplete Markets, Risk Entanglement, International Correlations, Asset Market View.

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1 Introduction

In frictionless and fully integrated international financial markets, the absence of arbitrage opportunities implies an equality between the exchange rate and the ratio of stochastic discount factors (SDFs) of the two countries when markets are complete. Because SDFs represent economic fundamentals and their risk pricing, the asset market view of the exchange rate determination adopts this equality to relate exchange rates to countries' pricing dynamics. When markets are incomplete, an ambiguity arises due to the multiplicity of SDFs. The asset market view *postulates* to employ the minimum-variance SDFs (a.k.a., SDF projectors) in the above equality, as these SDFs are unique and pure market-based constructs linear in asset returns. The postulated identification of the exchange rate with the ratio of SDF projectors is forthright but restrictive, leading to several strong asset pricing implications that are difficult to reconcile with international macro and price data.

In this paper, we first establish a necessary and sufficient condition on asset markets for the equality between the exchange rate and the ratio of SDF projectors. Our condition provides a complete and unambiguous classification of incomplete markets into two groups: the traditional asset market view of the exchange rate holds in the first, and it is rejected in the second. This complete characterization gives rise to a broader and more flexible version of the asset market view of the exchange rate determination that reconciles international price patterns. To the extent that the asset market view has been extensively employed to interpret the exchange rate dynamics from an asset pricing perspective, our finding emphasizes a crucial but subtle role of the market completeness in such an interpretation. In particular, asset prices are consistent with international macro data because exchange rate implications on the pricing can be substantially weakened by the incompleteness of asset markets.

Specifically, we prove that the exchange rate equals the ratio of SDF projectors *if and only if* every exchange rate risk can be separately contracted by trading assets, i.e., exchange rate risks are completely disentangled in asset markets. Therefore, it requires the *entanglement* of exchange rate risks, i.e., risks that impact the exchange rate but are not individually hedged in asset markets, to deviate the exchange rate from the ratio of SDF projectors.[1] This deviation relaxes the restrictive

[1]The formal definition of the exchange rate risk entanglement in a continuous-time setting is as follows (Definition 1). Suppose the exchange rate growth process consists of d diffusion and J jump risks. The exchange rate risk is entangled in asset markets when there exists at least a jump that cannot be individually replicated (hedged) by any portfolio of traded assets. In continuous time, jump risks and incomplete markets are necessary for exchange rate risks to be entangled in asset markets because (i) pure diffusion risks can always be rotated (redefined) and

asset pricing implications in the traditional asset market view by decoupling the exchange rate from countries' pricing dynamics. That is, in the presence of risk entanglement, a smooth exchange rate does not implicate a high correlation between countries' minimum-variance SDFs even when these SDFs are volatile.

Intuitively, risks are entangled in asset markets when they are collectively contracted in relatively few financial assets so that at least some of the risks are not individually traded. When exchange rate risks are entangled in FX markets, the asset sparsity enables a great flexibility in the mapping of risks into prices. As a result, observing prices of the same set of assets in all currency denominations does not suffice to unambiguously pin down the pricing of individual risks impacting these assets in any currency. Such an ambiguity facilitates a wedge between the exchange rate and ratio of SDF projectors, and therefore, mitigates the perplexity that implications of these two quantities appear to conflict in the data.

For a perspective, the approach in the current literature addressing this disparity is structural. By postulating and equipping the SDF with structural features, economic models can enrich and decouple the dynamics between SDFs, macro fundamentals and the exchange rate. Risk entanglement achieves this decoupling in a pure market-based approach. By no mean does it rule out structural explanations of international price patterns. Rather, risk entanglement sheds light on an important question that remains open in the elaborate structural framework: what is the relationship between the exchange rate and pricing dynamics (SDF projectors) when markets are incomplete? Risk entanglement addresses this question in generality. We show precisely that the absence of risk entanglement is both necessary and sufficient to identify the exchange rate with the ratio of SDF projectors. By implication, risk entanglement breaks this identity, and therefore, provides a possible justification for the counterfactuals that are deemed impossible under the traditional asset market view of the exchange rate.

Our study presents two practical aspects of the incomplete asset market view of the exchange rate. First, we introduce a measure to quantify the level of exchange rate risk entanglement in asset markets. The risk entanglement index captures the difference between exchange rate risk and the ratio of SDF projectors, both of which are market-based quantities. Because risk entanglement is the necessary and sufficient condition for this difference to arise, the risk entanglement index unambiguously implicates exchange rate risk entanglement in FX markets. To the extent that

completely disentangled in all asset markets (Section 3), and (ii) risks of all types are always completely disentangled in complete markets (by definition).

returns and their conditional moments are observable, the risk entanglement index is observable.

Second, we address the international correlation pattern, i.e., the empirically observed disparity between smooth exchange rates and modest cross-country correlations of macro quantities, in a numerical setup. The setup features exchange rate risk entanglement, which rationalizes the disassociation of the exchange rate from pricing dynamics, and hence implies their disparity. Assuming frictionless international financial markets,[2] the strength of this approach is in the result that risk entanglement, as a necessary and sufficient condition, is the only possible pure market-based mechanism to disconnect the exchange rate from pricing dynamics. To stay true to the asset market view, our numerical setup does not employ elaborate structural features. Their addition to the model would strengthen the approach.

Our paper is related to the literature studying the exchange rate dynamics from a market-based perspective. The advantages of this approach is its model-free nature, employing the most basic structural assumptions and making use of observable prices and exchange rate data. Such an asset market view of the exchange rate determination starts with Frenkel (1976), Kouri (1976), Mussa (1976), and also Dornbusch (1976). Early analytical relationships between risk pricing dynamics and the exchange rate are obtained by Saa-Requejo (1994), Zapatero (1995), and Backus et al. (2001). However, Lewis (1996), Obstfeld and Rogoff (2000), Engel (2016) and references therein empirically show a perplexing trend of disconnections between exchange rate dynamics and macro fundamentals. Our paper generalizes and hence also relaxes the implications of this asset market view to the incomplete market setting of risk entanglement.

Brandt et al. (2006) articulate the puzzling aspects of the international correlation pattern. Bakshi et al. (2017) and Chabi-Yo and Colacito (forthcoming) also suggest that SDFs across countries are very similar and highly correlated. These papers do not assume a diffusion model as in Brandt et al. (2006) but still rely on the assumption of complete markets. An equilibrium literature addresses the pattern by employing various structural features, including time non-separable preference (Colacito and Croce (2011)), heterogeneous sizes in conjunction with non-tradable good consumptions of economies (Hassan (2013)), heterogeneous sizes in conjunction with imperfect substitutes of country-specific outputs but without non-tradable consumptions (Martin (2013)), habit formation (Stathopoulos (2017)), and rare disaster events (Farhi and Gabaix (2016)). Risk entanglement is a market-based approach, and does not contradict or rule out structural approaches.

[2] We assume no arbitrages, perfect integration, and no transaction costs in international financial markets.

Another literature addresses the puzzle by employing various market frictions in structural models, including market segmentation as in Alvarez et al. (2002), Chien et al. (2015), non-diversifiable risks as in Sarkissian (2003)), limited financial infrastructures as in Maggiori (2017), Gabaix and Maggiori (2015), and other frictions (e.g., Pavlova and Rigobon (2008), Favilukis et al. (2015)). While risk entanglement does not require segmentation and other frictions, it is an imperfection of incomplete asset markets.

Taking a different perspective, Burnside and Graveline (2012) question the equality between the exchange rate and ratio of SDF projectors, and consequently also question the asset market view of the exchange. The current paper establishes risk entanglement in the exchange rate as the necessary and sufficient condition for the above equality to break down. We thus clarify that Burnside and Graveline (2012)'s observations apply when exchange rate risks are entangled, while the traditional asset market view holds when exchange rate risks are completely disentangled.

Lustig and Verdelhan (2016) employ a parsimonious incomplete international market model suggested by Backus et al. (2001) to investigate three anomalies (namely the international correlation, Backus-Smith, and currency premium patterns). They characterize analytically in log-normal settings, and numerically beyond log-normal settings, the difficulty to match all three anomalies. Bakshi et al. (forthcoming) and Sandulescu et al. (2017) estimate SDFs and quantify the three puzzles if markets are segmented. Our current paper examines only the international correlation puzzle as an application and works with fully integrated markets. The paper instead focuses principally on formal aspects of an incomplete asset market view of the exchange rate, employing exclusively the projection methodology. Maurer and Tran (2016) offer a different risk entanglement perspective on generating multiple exchange rates from given structural SDFs, and through this flexibility address the three anomalies above by settling on an empirically relevant exchange rate solution. Whereas the setup therein directly accommodates structural inputs, because of this modeling feature, it does not establish a pure market-based view of the exchange rate. Only the SDF projector construction, which relies solely on asset return inputs and is employed in the current paper, explicitly enforces the pure market content and substantiates this view in incomplete markets.

Looking beyond FX markets and the associated quantitative puzzles, our current paper contributes to an important if technically challenging literature to quantify the role of incomplete asset markets in the relationship between prices and quantities.

The current paper is organized as follows. Section 2 motivates the basics for the exchange rate risk entanglement in an intuitive discrete setting and elaborates details on the applicability of Burnside and Graveline (2012)'s findings. Section 3 formalizes our setup to study the incomplete asset market view of the exchange rate determination and presents the main results of the paper. Section 4 introduces the risk entanglement index and calibrates the international correlation pattern. Section 5 concludes. Appendix A details the construction of SDF projectors and notation employed in the main text. Finally, Appendix B derives main results of the paper and offers further technical insights into the concept of risk entanglement.

2 Asset Market View of the Exchange Rate: Numerical Illustrations

Before presenting a general discussion and technical aspects of the asset market view of the exchange rate when markets are incomplete, we illustrate the key underlying relationship between the exchange rate and the ratio of SDF projectors in two simple numerical examples. We demonstrate the upholding of the asset market view in the first setting, and its failure in the second.[3]

Assuming arbitrage-free, integrated and frictionless international financial markets, our illustration proceeds as follows. We take as given (i) asset returns in the home currency denomination, and (ii) the exchange rate process in an incomplete market. As international financial markets are fully integrated, every asset traded in the home market is also traded in the foreign market (and vice versa), and asset returns in the foreign currency denomination are those in the home currency denomination multiplied by the exchange rate factor.[4] Knowing asset returns in home and foreign currency denominations, we construct the unique minimum-variance SDFs for home and foreign countries. These SDFs are linear in asset returns denominated in respective currencies, and are also known and referred to as SDF projectors (Hansen and Jagannathan (1991)). Finally, we examine the relationship between the ratio of constructed SDF projectors and the exchange rate input. An equality (inequality) indicates the upholding (failure) of the asset market view of the exchange rate determination. All supporting derivations are relegated to Appendix A. General results in continuous settings are presented in subsequent sections.

[3]The specific examples and their inputs are among the simplest possible just to illustrate the upholding and failure of the characteristic equation of the asset market view of the exchange rate determination. Hence these input values are not intended to match empirical counterparts. A more realistic calibration is relegated to Section 4.

[4]Evidently, this change of denominations applies either markets are complete or incomplete.

state	EXrate State	EXrate	Asset Returns in currency H				H's Proj.	Asset Returns in currency F				F's Proj.	Proj. Ratio
s	S	e	B_H	$\frac{B_F}{e}$	Y_1	Y_2	$M_{H\|}$	eB_H	B_F	eY_1	eY_2	$M_{F\|}$	$\frac{M_{H\|}}{M_{F\|}}$
1		0.95	1.01	1.09	0	1	0.099	0.96	1.04	0	0.95	0.105	0.95
2	S_1	0.95	1.01	1.09	0	4	1.022	0.96	1.04	0	3.80	1.076	0.95
3		0.95	1.01	1.09	0	2	0.407	0.96	1.04	0	1.90	0.428	0.95
4	S_2	1.2	1.01	0.87	0	0	2.589	1.21	1.04	0	0	2.158	1.2
5	S_3	0.8	1.01	1.3	6	0	0.833	0.81	1.04	4.80	0	1.042	0.8

Table 1: Upholding the Asset Market View of the Exchange Rate Determination, $e(s) = \frac{M_{H\|}(s)}{M_{F\|}(s)}$, $\forall s \in \{1, \ldots, 5\}$: (i) all future states $s \in \mathcal{S} = \{1, 2, 3, 4, 5\}$ have same probability of 0.2, (ii) asset returns and exchange rate are given in term of their gross growths, (iii) risk-free rates $r_H = 0.01$ and $r_F = 0.04$. SDF projectors are computed as linear functions of asset returns in respective currency denominations, with numerical solutions for coefficients as follows, $\{\beta_H = 13.09, \beta_F = -12.27, \beta_1 = 0.59, \beta_2 = 0.31\}$ for $M_{H\|}$, and $\{\widehat{\beta}_H = 9.41, \widehat{\beta}_F = -8.90, \widehat{\beta}_1 = 0.56, \widehat{\beta}_2 = 0.34\}$ for $M_{F\|}$.

2.1 Confirming the Asset Market View

To illustrate the upholding of the asset market view of the exchange rate determination, we consider a one-period (from t to $t+1$) market setting of four non-redundant assets, and five future states $s \in \mathcal{S} \equiv \{1, \ldots, 5\}$ to be realized at $t+1$ with equal probabilities $p(s) = 0.2$. Evidently, financial markets are incomplete because there are not enough assets to hedge every risk state. Our convention for the exchange rate quotes the amount of e units of the foreign currency that buys one unit of the home currency. The current exchange rate at t is normalized to one. The set of possible realizations $\{e(s)\}$ of the future exchange rate at $t+1$ is given in the setting, which is also the set of realized exchange rate gross growths in our normalization.

The four asset returns are given in the home currency denomination, namely the gross returns $B_H = 1 + r_H$ and $\frac{B_F}{e(s)} = \frac{1+r_F}{e(s)}$ respectively on home and foreign bonds, and the gross returns $Y_1(s)$, $Y_2(s)$ on two other assets, where r_H and r_F are home and foreign risk-free rates.[5] We assume values $r_H = 0.01$ and $r_F = 0.04$ for risk-free rates. Table 1 (columns 3 through 7) reports numerical inputs to our setting.

Based on these asset return inputs, we next construct unique home and foreign SDF projectors that (i) are linear in asset returns, and (ii) price asset returns in the respective currencies. The

[5] In the home currency denomination, while the gross return on the home risk-free bond $B_H = 1+r_H$ is independent of state s, the gross return on the foreign risk-free bond $\frac{B_F}{e(s)} = \frac{1+r_F}{e(s)}$ varies with state s via the exchange rate factor $e(s)$.

SDF projector has minimum variance among all pricing-consistent SDFs (Hansen and Jagannathan (1991)). Specifically, the home SDF projector $M_{H\|}(s)$ is determined by solving for coefficients $\{\beta_H, \beta_F, \beta_1, \beta_2\}$ in the linear construct, $M_{H\|}(s) = \beta_H B_H + \beta_F \frac{B_F}{e(s)} + \beta_1 Y_1(s) + \beta_2 Y_2(s)$, that prices all four returns in the home currency,[6]

$$E_t\left[M_{H\|} B_H\right] = 1, \quad E_t\left[M_{H\|}\frac{B_F}{e}\right] = 1, \quad E_t\left[M_{H\|} Y_1\right] = 1, \quad E_t\left[M_{H\|} Y_2\right] = 1.$$

These four pricing equations constitute a linear system which uniquely determines the four β coefficients, and thus, the home SDF projector $M_{H\|}(s)$. We report numerical solutions of these quantities in Table 1 and its caption, and relegate further details and derivation to Appendix A.1.

Because international markets are fully integrated, foreign investors trade these same four basis assets (and their portfolios). Therefore, asset returns to foreign investors are spanned by these four basis asset returns denominated in the foreign currency, namely $\{B_H e(s), B_F, Y_1(s)e(s), Y_2(s)e(s)\}$. A similar procedure then determines uniquely the foreign SDF projector $M_{F\|}(s) = \widehat{\beta}_H B_H e(s) + \widehat{\beta}_F B_F e(s) + \widehat{\beta}_1 Y_1(s)e(s) + \widehat{\beta}_2 Y_2(s)e(s)$ from the returns denominated in the foreign currency (of the same four assets). We also report numerical solutions of the foreign SDF projector $M_{F\|}(s)$ and the associated $\widehat{\beta}$ coefficients in Table 1 and its caption.

Before proceeding to verify the asset market view of the exchange rate determination for the setting of Table 1, it is helpful to briefly describe the basics of such a view in incomplete markets (see e.g., Brandt et al. (2006)). First, when markets are complete, there exists a unique SDF M_I for country $I \in \{H, F\}$ that is also identical to the SDF projector $M_{I\|}$. The no arbitrage condition then implies that the exchange rate equals the ratio of two SDFs, $e(s) = \frac{M_H(s)}{M_F(s)}$, $\forall s$. Next, when markets are incomplete, there exist multiple SDFs that price asset returns. It therefore appears natural to substitute the unobserved SDFs by their market-based and unique projectors. The asset market view of the exchange rate determination when markets are incomplete then is quantified by a postulated state-by-state identity,

$$e(s) = \frac{M_{H\|}(s)}{M_{F\|}(s)}, \qquad \forall s \in \mathcal{S}. \tag{1}$$

However, incomplete markets may also give rise to an apparent contradiction of such a view. Indeed,

[6]Technically, because B_H, $\frac{B_F}{e}$, Y_1, Y_2 are gross returns on assets, $M_{H\|}(s)$ here denotes the gross growth of SDF projected on the space of asset gross returns.

the substitution of linear projectors into (1) produces a counter-intuitive identity,

$$\begin{aligned} e(s) &= \frac{\beta_H B_H + \beta_F \frac{B_F}{e(s)} + \sum_n \beta_n Y_n(s)}{\widehat{\beta}_H B_H e(s) + \widehat{\beta}_F B_F + \sum_n \widehat{\beta}_n Y_n(s) e(s)} \\ &= \frac{1}{e(s)} \frac{\beta_H (1+r_H) e(s) + \beta_F (1+r_F) + \sum_n \beta_n Y_n(s) e(s)}{\widehat{\beta}_H (1+r_H) e(s) + \widehat{\beta}_F (1+r_F) + \sum_n \widehat{\beta}_n Y_n(s) e(s)}, \end{aligned} \qquad \forall s \in \mathcal{S}. \tag{2}$$

In particular, Burnside and Graveline (2012) observe that the exchange rate $e(s)$ factors in the two sides of (2) very differently. Following from this observation, they posit the impossibility of this characteristic identity (2) and the validity of the asset market view of the exchange rate determination when markets are incomplete.

Qualitatively, we observe that there exist subtle features underlying (2) that might help to uphold that identity for some classes of incomplete markets. That is, e.g., while the exchange rate and asset returns denominated in the home currency are largely mutually exogenous in the setup, the foreign SDF projector and its $\widehat{\beta}$ coefficients are endogenous to the exchange rate specification $\{e(s)\}$. This dependence of $\widehat{\beta}$ on $\{e(s)\}$ complicates the right-hand side of (2), and as a result, how the ratio of SDF projectors varies with the exchange rate is not obvious.[7]

Quantitatively, by design, our setup enables an explicit verification (or rejection) of identity (1) by directly comparing the given exchange rate with the ratio of SDF projectors constructed above from the given asset returns as suggested by (2). For the specific incomplete market setting under consideration, Table 1 illustrate such a direct verification of the asset market view of the exchange rate determination.

Clearly in Table 1, columns 3 and 14 are identical, i.e., the postulated equality $e(s) = \frac{M_{H\|}(s)}{M_{F\|}(s)}$ (1) is satisfied for every state $s \in \mathcal{S}$. In the sense of this equality, the asset market view of the exchange rate determination holds for the market configuration under consideration.

Discussion

The incomplete asset market configuration reported in Table 1 enables the characteristic identity $e = \frac{M_{H\|}}{M_{F\|}}$. It hence constitutes an explicit counterexample to the Burnside and Graveline (2012)'s impossibility result, discussed below (2), that the exchange rate does not equal the ratio of SDF

[7]We remark below that incomplete markets can render return spaces (of the same set of assets) in home and foreign currency denominations either identical or distinct. We then prove that identity (2) holds in the former, and fails in the latter case (Theorem 2).

projectors when asset markets are incomplete. We alleviate this impossibility result by identifying FX markets that can be decomposed and orthogonalized by the exchange rate risks. We elaborate on this upholding of the asset market view of the exchange rate determination in Table 1 via several general observations.

First and importantly, exchange rate risk can be completely hedged in the asset market configuration in Table 1. To see this, we observe that the state space $\mathcal{S}$ under consideration can be partitioned into three (composite) states $S_1 \equiv \{1,2,3\}$, $S_2 \equiv \{4\}$, $S_3 \equiv \{5\}$ that are distinguishable by the exchange rate. Accordingly, we refer to states in $\{S_1, S_2, S_3\}$ as exchange rate states. Note that the three returns B_H, $\frac{B_F}{e}$, Y_1 are adapted precisely to the partition $\{S_1, S_2, S_3\}$.[8] Therefore, to every exchange rate state S_i $(i \in \{1,2,3\})$ we can construct a corresponding portfolio (i.e., Arrow-Debreu asset) AD_i $(i \in \{1,2,3\})$ of traded assets $\left\{B_H, \frac{B_F}{e}, Y_1\right\}$ that pays off in and only in state S_i. The explicit composition of AD assets in term of original assets $\{B_H, \frac{B_F}{e}, Y_1\}$ (and the reverse composition of original assets in term of AD assets) are as follows,

$$\begin{bmatrix} AD_1 \\ AD_2 \\ AD_3 \end{bmatrix} = \begin{bmatrix} -12.31 & 14.35 & -1.04 \\ 9.18 & -8.47 & 0.29 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} B_H \\ \frac{B_F}{e} \\ Y_1 \end{bmatrix}; \quad \begin{bmatrix} B_H \\ \frac{B_F}{e} \\ Y_1 \end{bmatrix} = \begin{bmatrix} 0.31 & 0.52 & 0.17 \\ 0.33 & 0.45 & 0.22 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} AD_1 \\ AD_2 \\ AD_3 \end{bmatrix}. \tag{3}$$

For such a construction, AD assets $\{AD_1, AD_2, AD_3\}$ indeed pay off uniformly in and only in the respective exchange rate states $\{S_1, S_2, S_3\}$,

$$AD_1 = \begin{bmatrix} 3.27 \\ 3.27 \\ 3.27 \\ 0 \\ 0 \end{bmatrix}, \quad AD_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1.93 \\ 0 \end{bmatrix}, \quad AD_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 6 \end{bmatrix}, \tag{4}$$

In this sense, indeed the exchange rate risk characterized by exchange rate states $\{S_1, S_2, S_3\}$ is completely (and separately) contracted by trading Arrow-Debreu (AD) assets $\{AD_1, AD_2, AD_3\}$. We will formalize and refer to this key property that each exchange rate state can be singly contracted in asset markets as the complete disentanglement of exchange rate risks in Section 3 below.

[8] That is, return variables B_H, $\frac{B_F}{e(s)}$, $Y_1(s)$ are measurable with respect to the $\sigma-$algebra generated by the exchange rate variable $e(s)$.

Second, the space of asset returns can be partitioned into orthogonal subspaces associated with exchange rate states. Indeed, by construction, the original returns $\left\{B_H, \frac{B_F}{e}, Y_1\right\}$ can be equivalently transformed to AD asset returns $\{AD_1, AD_2, AD_3\}$ which are pairwise orthogonal (i.e., uncorrelated) and non-zero only in sub-states of respective exchange rate states $\{S_1, S_2, S_3\}$. The remaining return on Y_2 is non-zero only in sub-states $\{1, 2, 3\}$ of S_1, hence belongs to the orthogonal subspace associated with S_1. Asset Y_2's presence aims to demonstrate that the entire asset return space is not necessarily fully characterized by the exchange rate alone and asset markets are genuinely incomplete.[9] By construction, the return space in the home currency denomination is linearly spanned by either the original asset returns $\left\{B_H, \frac{B_F}{e}, Y_1, Y_2\right\}$ or the equivalent basis $\{AD_1, AD_2, AD_3, Y_2\}$. Using $\mathcal{R}$ to denote the linear spanning of asset returns, we symbolically express the equivalence of these two bases (4) as $\mathcal{R}\left(B_H, \frac{B_F}{e}, Y_1, Y_2\right) = \mathcal{R}\left(AD_1, AD_2, AD_3, Y_2\right)$. We summarize the orthogonalization and mapping of the asset returns $\{AD_1, AD_2, AD_3, Y_2\}$ into the space of exchange rate risk states $\{S_1, S_2, S_3\}$ as follows,

$$\{AD_1, Y_2\} \longrightarrow S_1 = \{1, 2, 3\}; \qquad \{AD_2\} \longrightarrow S_2 = \{4\}; \qquad \{AD_3\} \longrightarrow S_3 = \{5\}. \tag{5}$$

The mapping indicates that assets only pay off non-zero in the corresponding exchange rate state (and its sub-states) that they are mapped into (Table 1).

Third, the asset market view of the exchange rate determination holds for the asset market configuration of Table 1 because the key identity (2) holds separately for every orthogonal subspace of asset return space (5). For concreteness, let us examine the identity (2) at a state $s \in \mathcal{S}$ that belongs to exchange rate state S_i. Then, only assets paying off in S_i contribute to the right-hand side of (2) evaluated at state s under consideration (because assets associated with any other exchange rate states $S_i' \neq S_i$ have zero payoffs in s). By the same reason, for all sub-states of each exchange rate state S_i, the identity (2) involves only assets in the orthogonal return subspace associated with S_i. Because the exchange rate $e(s)$ does not vary with sub-states s in S_i, it reduces to a mere constant in the relationship (2) in subspace S_i. Accordingly, identity (2) holds trivially for all sub-states in each S_i, even when asset markets are incomplete with respect to these sub-states.[10] Specifically, the asset market configuration of Table 1 upholds (2) by separately satisfying

[9] Note that Y_2's return varies with sub-states $\{1, 2, 3\}$ of S_1. In the absence of Y_2, asset return states (risks) in Table 1 are effectively reduced to exchange rate states (risks) because no traded assets can distinguish original states of $\{1, 2, 3\}$ of S_1. Asset markets are effectively complete in absence of Y_2.

[10] That the exchange rate does not vary with sub-states $\{1, 2, 3\}$ of S_1 mitigates all effects stemming from the variation of return Y_2 with these sub-states, and preserves identity (2) in these sub-states.

three identities for the exchange rate states,[11]

$$\left.\frac{M_{H\|}(s)}{M_{F\|}(s)}\right|_{s\in S_1=\{1,2,3\}} = \left.\frac{(0.31\times\beta_H+0.33\times\beta_F)\times AD_1(s)+\beta_2\times Y_2(s)}{\left[(0.31\times\widehat{\beta}_H+0.33\times\widehat{\beta}_F)\times AD_1(s)+\widehat{\beta}_2\times Y_2(s)\right]\times e(s)}\right|_{s\in S_1=\{1,2,3\}} = 0.95 = e(S_1),$$

$$\left.\frac{M_{H\|}(s)}{M_{F\|}(s)}\right|_{s\in S_2=\{4\}} = \left.\frac{[0.52\times\beta_H+0.45\times\beta_F]\times AD_2(s)}{[0.52\times\widehat{\beta}_H+0.45\times\widehat{\beta}_F]\times AD_2(s)\times e(s)}\right|_{s\in S_2\equiv\{4\}} = 1.2 = e(S_2) = e(4), \tag{6}$$

$$\left.\frac{M_{H\|}(s)}{M_{F\|}(s)}\right|_{s\in S_3=\{5\}} = \left.\frac{[0.17\times\beta_H+0.22\times\beta_F+\beta_1]\times AD_3(s)}{[0.17\times\widehat{\beta}_H+0.22\times\widehat{\beta}_F+\widehat{\beta}_1]\times AD_3(s)\times e(s)}\right|_{s\in S_3\equiv\{5\}} = 0.8 = e(S_3) = e(5).$$

Finally, the above orthogonalization of the asset return space by the exchange rate risk (5) implies the invariance of return spaces to the currency denomination in Table 1. In fact, for the asset market configuration under consideration, this invariance is reflected in the fact that asset return bases $\left\{B_H, \frac{B_F}{e}, Y_1, Y_2\right\}$ (in the home currency denomination) and $\{eB_H, B_F, eY_1, eY_2\}$ (in the foreign currency denomination) span identical spaces.[12] As a result, SDF projectors $M_{H\|}$ and $M_{F\|}$, which are linear constructs in asset returns in respective denomination, belong to the same space. They can be matched state-by-state by the exchange rate factor to uphold the characteristic identity (1) of the asset market view, albeit markets are incomplete.

In Section 3, we fully generalize these observations on the specific market configuration of Table 1 in continuous settings. We establish that every exchange rate risk being separately contracted in asset markets, i.e., complete disentanglement of the exchange rate risk (Definition 1), is a necessary and sufficient condition to enforce the asset market view of the exchange rate determination in incomplete markets (Theorems 1). We also show that the asset market view holds if and only if asset return spaces in home and foreign currency denominations are identical (Theorems 2).

2.2 Rejecting the Asset Market View

To illustrate the failure of the asset market view of the exchange rate determination, we now consider an alternative incomplete asset market setting. The observations and discussion above

[11] On one hand, the explicit ratio of SDF projectors in (6) is computed by substituting original asset returns $\{B_H, \frac{B_F}{e}, Y_1\}$ by orthogonal returns $\{AD_1, AD_2, AD_3,\}$ (while Y_2 is kept intact) in projectors $M_{H\|}$, $M_{F\|}$, using (3) and β and $\widehat{\beta}$ coefficients listed in the caption of Table 1. On the other hand, the numerical values $\{e(s)\}$ of the exchange rate are specified in the setting. The verified matching of the SDF projector ratio with these numerical values in (6) then demonstrates the equality $\frac{M_{H\|}(s)}{M_{F\|}(s)} = e(s)$ for every state $s \in \mathcal{S} = \{1, \ldots, 5\}$.

[12] The orthogonality (5) assures the following equivalent relationships between return bases (and the return spaces generated by them): $\{eB_H, B_F, eY_1, eY_2\} \iff \{eAD_1, eAD_2, eAD_3, eY_2\} \iff \{AD_1, AD_2, AD_3, Y_2\} \iff \left\{B_H, \frac{B_F}{e}, Y_1, Y_2\right\}$.

state	EXrate State	EXrate	Asset Returns in currency H				H's Proj.	Asset Returns in currency F				F's Proj.	Proj. Ratio
s	S	e	B_H	$\frac{B_F}{e}$	Y_1	Y_2	$M_{H\parallel}$	eB_H	B_F	eY_1	eY_2	$M_{F\parallel}$	$\frac{M_{H\parallel}}{M_{F\parallel}}$
1		0.95	1.01	1.09	2	1	1.032	0.96	1.04	1.9	0.95	0.932	1.11
2	S_1	0.95	1.01	1.09	0	4	0.835	0.96	1.04	0	3.80	0.899	0.93
3		0.95	1.01	1.09	0	2	0.315	0.96	1.04	0	1.90	0.368	0.86
4	S_2	1.2	1.01	0.87	0	0	2.280	1.21	1.04	0	0	1.936	1.18
5	S_3	0.8	1.01	1.3	6	0	0.489	0.81	1.04	4.80	0	0.673	0.73

Table 2: Rejecting the Asset Market View of the Exchange Rate Determination, $e(s) \neq \frac{M_{H\parallel}(s)}{M_{F\parallel}(s)}$, $\forall s \in \{1,\ldots,5\}$: (i) all future states $s \in \mathcal{S} = \{1,2,3,4,5\}$ have same probability of 0.2, (ii) asset returns and exchange rate are given in term of their gross growths, (iii) risk-free rates $r_H = 0.01$ and $r_F = 0.04$. SDF projectors are computed as linear functions of asset returns in respective currency denominations, with numerical solutions for coefficients as follows, $\{\beta_H = 11.60, \beta_F = -10.89, \beta_1 = 0.49, \beta_2 = 0.26\}$ for $M_{H\parallel}$; and $\left\{\widehat{\beta}_H = 8.32, \widehat{\beta}_F = -7.83, \widehat{\beta}_1 = 0.44, \widehat{\beta}_2 = 0.28\right\}$ for $M_{F\parallel}$.

suggest to annul the key feature of the market configuration of Table 1, namely every exchange rate state is separately contracted in asset markets. There are many specific ways to achieve this annulment (see also our discussion below). For a simple illustration, we replace asset return Y_1 in Table 1 by another asset Y_1' which pays off in both states $s = 1$ and $s = 5$. All other inputs and notation remain unchanged from Table 1. Model outputs (SDF projectors and their ratio) are computed in an identical approach. Table 2 reports the inputs and outputs of the current asset market configuration.

Compared to market inputs of Table 1, the only difference in the asset market inputs of Table 2 is the payoff of asset Y_1' in state $s = 1$, which is $Y_1'(1) = 2$ (compared to $Y_1(1) = 0$ in Table 1). Clearly, this simple modification in asset market configuration suffices to reject the postulated equality $e(s) = \frac{M_{H\parallel}(s)}{M_{F\parallel}(s)}$ (1) for every state $s \in \mathcal{S}$ in Table 2. In the sense of this rejection, the asset market view of the exchange rate determination fails for the market configuration under consideration.

Discussion

The incomplete asset market configuration reported in Table 2 disqualifies the characteristic identity $e = \frac{M_{H\parallel}}{M_{F\parallel}}$. It hence constitutes an explicit example to adhere to Burnside and Graveline (2012)'s impossibility result that the exchange rate does not equal the ratio of SDF projectors when asset markets are incomplete. We adhere to this impossibility result by identifying FX markets that cannot be decomposed and orthogonalized by the exchange rate risk. We elaborate on this rejection of the asset market view of the exchange rate determination in Table 2 via several observations, which mirror those made earlier in Table 1.

First, the exchange rate risk cannot be completely hedged in the asset market configuration in Table 2. To see this, note that the exchange rate states remain $S_1 \equiv \{1,2,3\}$, $S_2 \equiv \{4\}$, $S_3 \equiv \{5\}$.[13] However, there does not exist a portfolio (Arrow-Debreu asset) AD_i of traded assets $\left\{B_H, \frac{B_F}{e}, Y_1', Y_2\right\}$ that pays off non-zero and identically in (and only in) sub-states of S_i, for all $i \in \{1,2,3\}$. Therefore, none of the exchange rate states (i.e., risk) can be separately contracted in asset markets. We will formalize and refer to this key property as the entanglement of exchange rate risks in Section 3 below. In retrospect, other choices of asset returns that do not give rise to a complete set of AD assets $\{AD_i\}$ (each pays off non-zero and identically only in sub-states of respective S_i) will also illustrate the failure of identity (1).

Second and furthermore, the space of asset returns cannot be partitioned into orthogonal subspaces associated with exchange rate states. As a result, the postulated identity (2) cannot be adapted separately to subspaces of asset returns of the form (6). In fact, for every state $s \in \mathcal{S}$, the right-hand side of (2) involves traded assets that pay off in different exchange rate states S_i. Therefore, identity (2) evaluated at sub-states s belonging to different exchange rate states S_i are related to one another via the asset returns, i.e., exchange rate risks are entangled in asset markets. The exogeneity between the specification of asset returns and the specification of the exchange rate in the setting then prevents an equality between the two sides of (2) as observed by Burnside and Graveline (2012).

Finally, there is a dependence of the asset return space on the currency denomination in Table 2. In fact, for the asset market configuration under consideration, this dependence is reflected in the fact that asset return bases $\left\{B_H, \frac{B_F}{e}, Y_1', Y_2\right\}$ (in the home currency denomination) and $\{eB_H, B_F, eY_1', eY_2\}$ (in the foreign currency denomination) span different spaces. As a result, SDF projectors $M_{H\|}$ and $M_{F\|}$, which are linear constructs in asset returns in respective denomination, belong to different spaces. They cannot be matched state-by-state by the exchange rate factor to uphold the characteristic identity (1) of the asset market view.

In Section 3, we fully generalize these observations on the specific market configuration of Table

[13] Because the exchange rate variable remains the same in Tables 1 and 2.

2 in continuous settings. We establish that the existence of some exchange rate risk that cannot be separately contracted in asset markets, i.e., exchange rate risk entanglement (Definition 1), is a necessary and sufficient condition to reject the asset market view of the exchange rate determination in incomplete markets (Theorems 1). We also show that asset return spaces in home and foreign currency denominations being different is equivalent to the rejection of such a view (Theorems 2).

3 General Setup and Main Results

This section formalizes our main findings on the incomplete asset market view of the exchange rate determination based upon the intuitive insights of the numerical examples. We also quantify the key underlying concept of risk entanglement in FX markets.

3.1 Setup

Our setup is in continuous time and mirrors that in the discrete setting of Section 2. We focus on two countries $I \in \{H, F\}$ in an international economy, and proceed from the perspective of home investors.

Assumptions: We make the following basic assumptions,

- A1 *The international financial markets are arbitrage-free, frictionless and fully integrated.*
- A2 *Country-specific risk-free bonds for every country are available as traded assets in financial markets.*

Assumption A1 simply asserts that if home investors can trade an asset, foreign investors can also trade that same asset without transaction costs, and vice versa.[14] It is a property of open and frictionless market economies. Assumption A2 is innocuous; the availability and tradability of risk-free bonds are the basis of currency carry trades in FX markets. These assumptions attest to open and integrated international financial markets as we see in developed markets today. However, the assumptions do not rule out the possibility of market incompleteness.

In our analysis, the key point of these assumptions is that even under these idealistic market conditions, the unity of the exchange rate and the ratio of SDF projectors – or the asset market

[14] Returns in home and foreign currency denominations (on the same asset), which are respectively earned by home and foreign investors, differ only by the exchange rate factor.

view of the exchange rate determination – is not warranted. Weakening these assumptions, therefore, will only strengthen this finding.

Inputs: In the asset market approach, we take as given only the price characteristics of international financial markets. Specifically, the exogenous inputs to the setting are (i) the exchange rate process, and (ii) return distributions denominated in the home currency of the entire universe of traded assets.

(i) Exchange Rate Specification: We take the exchange rate e_t (i.e., the amount of the foreign currency that buys one unit of the home currency at time t) as a given jump-diffusion process,

$$\frac{e_{t+dt}}{e_t} = 1 + \mu_e dt + \sigma_e^T dZ_t + \delta_e^T d\mathcal{N}_t, \quad (7)$$

where μ_e denotes the drift term of the exchange rate growth, $Z_t \in \mathbf{R}^d$ a d-dimensional standard Brownian motion (i.e., d diffusion risks), and $\sigma_e \in \mathbf{R}^d$ the associated diffusion volatility vector. There is a set $\mathcal{J} = \{1, \dots, J\}$ of J different jump risks in the world economy. We assume that jump risks of different types are independent, and are represented by J standard independent Poisson processes $\mathcal{N}_t = \{\mathcal{N}_{it}\}_{i \in \mathcal{J}}$ of respective arrival intensities in vector $\lambda \equiv \{\lambda_i\}_{i \in \mathcal{J}}$. The associated jump sizes in the exchange rate growth are in given in vector $\delta_e \equiv \{\delta_{ei}\}_{i \in \mathcal{J}}$. That is, the probability that a jump of type $i \in \mathcal{J}$ takes place in an infinitesimal time interval from t to $t + dt$ is $\lambda_i dt$. Following a jump of type i, the exchange rate growth immediately increases by δ_{ei}, $i \in \mathcal{J}$. If the exchange rate is not subject to the jump of type i, we set the associated jump size to zero, $\delta_{ei} = 0$. We refer to Appendix A.2 for a complete notational description. Finally, we assume that exchange rate's jump sizes are different for different types of jump risks, and have significant magnitudes.[15]

(ii) Asset Return Specification: Because international financial markets are fully integrated (Assumption A1), every investor in the world trades the same universe of financial assets, which can either originating from home or foreign economies. Returns on these assets to an (home or foreign) investor, however, are denominated in the (home or foreign) currency of the investor's country.

[15] That is, first, if jumps of different types i and j impact the exchange rate $(\delta_{ei}, \delta_{ej} \neq 0)$, then $\delta_{ei} \neq \delta_{ej}$. Second, jumps affect the per-home-currency exchange rate e_t and per-foreign-currency exchange rate $\frac{1}{e_t}$ distinctly: $\delta_{ei} \neq -\delta_{\frac{1}{e}i}$, $\forall i \in \mathcal{J}$ s.t. $\delta_{ei} \neq 0$. These are technical assumptions needed to make sure that asset markets can differentiate all types of jump risks to the exchange rate, and jump and diffusion risks are distinct in magnitudes.

Without loss of generality, we first consider these asset returns in the home currency denomination, i.e., from home investors' perspective.

Let us consider a set of $N+1$ non-redundant basis asset returns of the return space on traded assets $\mathcal{R}$.[16] Specifically, the $N+1$ basis assets include the home risk-free bond, the foreign risk-free bond (Assumption A2 on the tradability of bonds), and $N-1$ other risky assets. Let gross returns on the assets follow (compensated) jump-diffusion processes in the home currency denomination,[17]

$$\frac{B_{Ht+dt}}{B_{Ht}} = 1 + \frac{dB_{Ht+dt}}{B_{Ht}} = 1 + r_H dt,$$

$$\frac{\frac{B_{Ft+dt}}{e_{t+dt}}}{\frac{B_{Ft}}{e_t}} = 1 + \frac{d\frac{B_{Ft+dt}}{e_{t+dt}}}{\frac{B_{Ft}}{e_t}} = 1 + \mu_{BF} dt + \sigma_{BF}^T dZ_t + \delta_{BF}^T \left(d\mathcal{N}_t - \lambda dt\right), \tag{8}$$

$$\frac{Y_{nt+dt}}{Y_{nt}} = 1 + \frac{dY_{nt+dt}}{Y_{nt}} = 1 + \mu_n dt + \sigma_n^T dZ_t + \delta_n^T d\mathcal{N}_t - \delta_n^T \lambda dt, \quad n \in \{1, \ldots, N-1\},$$

where r_H is the home risk-free rate, $\frac{B_{Ht+dt}}{B_{Ht}}$ the gross return on the home bond. $\frac{B_{Ft+dt}/e_{t+dt}}{B_{Ft}/e_t}$ is the gross return on the foreign bond denominated in the home currency,[18] whose moments μ_{BF}, σ_{BF} and jump sizes δ_{BF} are given in (38), Appendix A.2. $\frac{Y_{nt+dt}}{Y_{nt}}$ denotes the gross return on asset Y_n in the home currency denomination,[19] and $\mu_n \in \mathbf{R}$ denotes the mean, $\sigma_n \in \mathbf{R}^d$ the volatility vector, and $\delta_n \equiv \{\delta_{ni}\}_{i\in\mathcal{J}} \in \mathbf{R}^J$ the jump size vector of this return. The underlying diffusion and jump risks are as in the specification of the exchange rate process (below Equation (7) and Appendix A.2). In particular, if the return on asset Y_n is not subject to the jump of type i, we set the associated jump size to zero, $\delta_{ni} \equiv 0$. We relegate to Appendix A.2 further details on the notation.

Implied Quantities: In a setting of frictionless and integrated international financial markets (Assumptions A1, A2), the above inputs *uniquely* imply the following quantities; (i) the home SDF projected on the space of asset returns in the home currency (a.k.a, home SDF projector) $M_{H\|t}$, (ii) the asset returns in the foreign currency, and (iii) the foreign SDF projected on the space of asset returns in the foreign currency (a.k.a, foreign SDF projector) $M_{F\|t}$.

[16] That is, any traded asset return is linearly spanned by the returns on these $N+1$ basis assets. To be consistent with our earlier notation (and when this does not create ambiguities), $\mathcal{R}$ denotes interchangeably the set of $N+1$ basis returns as well as the asset space linearly spanned by them.

[17] Observe that, by convention, all asset returns in (8) have a compensated jump-diffusion form, i.e., the drift coefficient μ_n is also the expected return, $\mu_n = \frac{1}{dt} E_t \left[\frac{dY_{nt+dt}}{Y_t}\right]$. Thus μ_n contains compensations for asset Y_n's bearing of both diffusion and jump risks.

[18] Because in our exchange rate convention, e_t units of the foreign currency buys one unit of the home currency, $\frac{B_{Ft}}{e_t}$ is the time-t price of the foreign bond in the home currency.

[19] Recall that basis assets Y_n, $n \in \{1, \ldots, N-1\}$ may intrinsically originate from either home or foreign economies. Their return specification (8), however, is given in the home currency denomination in our setting's convention.

(i) Home SDF Projector: The unique home SDF projector $M_{H\|t}$ that prices and is linear in asset returns in the home currency is,

$$\frac{M_{H\|t+dt}}{M_{H\|t}} = 1 + \frac{dM_{H\|t+dt}}{M_{H\|t}} = 1 + \beta_H \frac{dB_{Ht+dt}}{B_{Ht}} + \beta_F \frac{d\frac{B_{Ft+dt}}{e_{t+dt}}}{\frac{B_{Ft}}{e_t}} + \sum_{n=1}^{N-1} \beta_n \frac{dY_{nt+dt}}{Y_{nt}}, \tag{9}$$

where β coefficients are derived in (53), Appendix A.2. This projector also has the minimum variance among all possible SDFs that price asset returns. As (9) indicates, $M_{H\|t}$ can be seen as in the projection of SDF net growth $\frac{dM_{Ht+dt}}{M_{Ht}}$ on the space of net asset returns in the home currency denomination.[20]

(ii) Asset Returns in the Foreign Currency Denomination: Since home and foreign investors trade the same set of assets (Assumption A1), asset returns in the foreign currency denomination are obtained from those in the home currency denomination (8) by the exchange rate multiplication. Specifically, the gross returns in the foreign currency denomination on the home bond, the foreign bond, and $N-1$ risky assets respectively are as follows,

$$\begin{aligned} \tfrac{e_{t+dt}B_{Ht+dt}}{e_t B_{Ht}} &= 1 + \mu_{BH}dt + \sigma_{BH}^T dZ_t + \delta_{BH}^T d\mathcal{N}_t - \delta_{BH}^T \lambda dt, \qquad \tfrac{B_{Ft+dt}}{B_{Ft}} = 1 + r_F, \\ \tfrac{Y_{Fnt+dt}}{Y_{Fnt}} = \tfrac{e_{t+dt}Y_{nt+dt}}{e_t Y_{nt}} &= 1 + \mu_{Fn}dt + \sigma_{Fn}^T dZ_t + \delta_{Fn}^T d\mathcal{N}_t - \delta_{Fn}^T \lambda dt, \quad n \in \{1, \ldots, N-1\}. \end{aligned} \tag{10}$$

The notation of these asset returns mirrors that of returns in the home currency denomination (8). An application of Ito's lemma yields explicit expressions for μ_{BH}, σ_{BH}, δ_{BH}, μ_{Fn}, σ_{Fn}, δ_{Fn} (see (36) and (37), Appendix A.2).

(iii) Foreign SDF Projector: After having obtained asset returns (10) in the foreign currency, the foreign SDF projector construction is similar to that of the home projector (9),

$$\frac{M_{F\|t+dt}}{M_{F\|t}} = 1 + \frac{dM_{F\|t+dt}}{M_{F\|t}} = 1 + \widehat{\beta}_H \frac{d(B_{Ht+dt}e_{t+dt})}{B_{Ht}e_t} + \widehat{\beta}_F \frac{dB_{Ft+dt}}{B_{Ft}} + \sum_{n=1}^{N-1} \widehat{\beta}_n \frac{d(Y_{nt+dt}e_{t+dt})}{Y_{nt}e_t}, \tag{11}$$

where returns on the right-hand side are in the foreign currency denomination of the home bond,

[20] We note that in continuous time, for consistency, the SDF projector construction necessarily involves net growth quantities (while in discrete time, either net or gross growth quantities produce a consistent SDF projector). See the discussion leading to Remark 1 in Appendix A.2.

the foreign bond, and risky assets respectively. The $\widehat{\beta}$ coefficients are as in (53) (with quantities referenced to the home currency replaced by their respective counterparts referenced to the foreign currency).

3.2 Key Hypothesis

Having presented a joint market-based setup of the exchange rate and asset returns, we state a hypothetical relationship to define and quantify the incomplete asset market view of the exchange rate determination.

Hypothesis H *In arbitrage-free, frictionless and perfectly integrated international financial markets (Assumptions A1-A2), the incomplete asset market view of the exchange rate determination is quantified by the equality between the exchange rate and the ratio of the two associated SDF projectors,*

$$\frac{e_{t+dt}}{e_t} \overset{!}{=} \frac{\frac{M_{H\|t+dt}}{M_{F\|t+dt}}}{\frac{M_{H\|t}}{M_{F\|t}}}, \quad \textit{or in simplified notation:} \quad e_t \overset{!}{=} \frac{M_{H\|t}}{M_{F\|t}}, \quad \forall t \in [0, \infty). \tag{12}$$

This hypothesis postulates a strong relationship between the exchange rate and asset pricing in each of the currencies for the asset market view of the exchange rate. It extends the known complete-market no-arbitrage relationship $e_t = \frac{M_{Ht}}{M_{Ft}}$ to incomplete markets, and is the continuous-setting version of relationship (1). The hypothesis represents a pure asset market view because (12) abstracts from the knowledge of structural SDFs of economies, replacing them with their projectors constructed exclusively and uniquely from asset returns.

When it holds, the hypothesis relates observable exchange rate dynamics with the asset pricing characteristics in each currency. It can be employed to evaluate international asset pricing models in the presence of incomplete markets (e.g., Brandt et al. (2006), and Section 4 below), in a manner similar to Hansen and Jagannathan (1991) bound tests.

Crucially, however, every application and implication of the important relationship (12) is subject to the validity of Hypothesis H itself. Our setup is designed to directly verify this hypothesis in the following procedure. We start with the returns (in the home currency denomination) of basis assets $\left\{B_H, \frac{B_F}{e}, \{Y_n\}\right\}$, $n \in \{1, \ldots, N-1\}$ (8) and the given exchange rate process e (7). These specifications imply uniquely (i) the home SDF projector $M_{H\|}$ (9), and (ii) asset returns (10) in the foreign currency and in turn, the foreign SDF projector $M_{F\|}$ (11). We then compare the exchange rate e with the ratio $\frac{M_{H\|}}{M_{F\|}}$. We present a complete analysis of Hypothesis H next.

3.3 Main Results: Risk Entanglements and the Asset Market View

Before stating the main result on the possible relationship between the exchange rate and SDF projectors, we formalize the concept of exchange rate risk entanglement. This risk entanglement concept is motivated and informed by our intuitive discussion of simple numerical examples at the end of Sections 2.1, 2.2. To illustrate this connection, we will also map the correspondence between market configurations in discrete settings (Tables 1, 2) and their counterparts in continuous settings.

Definition 1 (Exchange Rate Risk Entanglement)

1. *The exchange rate risks are entangled if there exist risks that impact the exchange rate's instantaneous growth $\frac{de_{t+dt}}{e_t}$ and are not separately contracted in asset markets.*

$$\exists i \in \mathcal{J} \text{ with } \delta_{ei} \neq 0 : \nexists P \in \mathcal{R}\left(B_H, \frac{B_F}{e}, \{Y_n\}_{n=1}^{N-1}\right) \text{ with } \frac{P_{t+dt}}{P_t} = 1+\mu_P dt+\delta_{Pi}\left(d\mathcal{N}_{it} - \lambda_i dt\right).$$

2. *Otherwise, the exchange rate risks are completely disentangled if every risk impacting the exchange rate's instantaneous growth $\frac{de_{t+dt}}{e_t}$ is separately contracted (i.e., individually traded) in asset markets.*

$$\forall i \in \mathcal{J} \text{ with } \delta_{ei} \neq 0 : \exists P \in \mathcal{R}\left(B_H, \frac{B_F}{e}, \{Y_n\}_{n=1}^{N-1}\right) \text{ with } \frac{P_{t+dt}}{P_t} = 1+\mu_P dt+\delta_{Pi}\left(d\mathcal{N}_{it} - \lambda_i dt\right).$$

In the above definition, $\mathcal{R}(\{X\})$ denotes the set of portfolios linearly spanned by assets $\{X\}$ (Footnote 16).

Drawing an analogy from our previous discussions in Sections 2.1 and 2.2, an exchange rate risk is separately contracted (i.e., individually traded) if there exists a traded portfolio that loads only on that risk. We observe that in continuous settings of only diffusion processes, exchange rate risks are always completely disentangled. This is because assets (as well as diffusion risks) can be linearly combined into portfolios which load singly on a diffusion risk of the exchange rate process. This observation indicates that jump risks in the exchange rate process are needed to give rise to risk entanglement and its diverse consequences in continuous settings.

Equipped with Definition 1, our main result establishes a necessary and sufficient condition for the asset market view of the exchange rate determination to hold in incomplete financial markets.

Theorem 1 (Incomplete Asset Market View of the Exchange Rate Determination)

In arbitrage-free, frictionless and perfectly integrated international financial markets (Assumptions A1-A2), the incomplete asset market view of the exchange rate determination is valid, i.e., Hypothesis H holds, if and only if exchange rate risks are completely disentangled in asset markets:

$$\text{Exchange rate risks are completely disentangled} \quad \Longleftrightarrow \quad e_t = \frac{M_{H\|t}}{M_{F\|t}}, \quad \forall t \in [0, \infty). \tag{13}$$

The derivation of this theorem is involved but also offers further insights on risk entanglement. We relegate a detailed proof to Appendix B. Crucially, Theorem 1 shows that the asset market view of the exchange rate determination does not hold universally. It further clearly and precisely identifies the premise under which Hypothesis H fails and the influential asset market view of the exchange rate needs to be revised. This premise concerns only risks in the exchange rate, but not other risks in financial markets.[21] Furthermore, the complete disentanglement of exchange rate risks is a currency-neutral concept.[22] Therefore, there is no need to make an explicit reference to a specific currency in the statement of Theorem 1.

The intuitive analysis of discrete settings (Section 2.2) has indicated that exchange rate risk entanglement is tantamount to the variation of the asset return space with the denomination currency. Theorem 2 below fully formalizes this intuition, and consequently, offers a second interpretation (and equivalent condition) for the incomplete asset market view of the exchange rate determination.

Theorem 2 (Incomplete Asset Market View of the Exchange Rate Determination 2)

In frictionless and perfectly integrated international financial markets (Assumptions A1, A2), the incomplete asset market view of the exchange rate determination is valid, i.e., Hypothesis H holds, if and only if the asset return space is invariant with the denomination currency:

$$\mathcal{R} = \mathcal{R}_{\mathcal{F}} \quad \Longleftrightarrow \quad e_t = \frac{M_{H\|t}}{M_{F\|t}}, \quad \forall t \in [0, \infty).$$

[21] Exchange rate risk entanglement is stronger than the notion of risk entanglement in asset markets discussed in Maurer and Tran (2016). Whereas entangled exchange rate risks imply risk entanglement in asset markets, the opposite is not necessarily true.

[22] Assuming A1, A2, if exchange rate risks are completely disentangled in one currency, they are completely disentangled in every currency.

Above, $\mathcal{R} = \left\{B_H, \frac{B_F}{e}, \{Y_n\}_{n=1}^{N-1}\right\}$ and $\mathcal{R_F} = \left\{eB_H, B_F, \{eY_n\}_{n=1}^{N-1}\right\}$ respectively denote the asset return space in home and foreign currency. A proof of this theorem is given in Appendix B.3. Theorem 2 offers a novel and equivalent characterization of the asset market view of the exchange rate by unambiguously relating this view to the possible mismatch of the two asset return spaces (in home and foreign currency denomination). This necessary and sufficient characterization is intuitive. On one hand, when asset return spaces in home and foreign currencies are identical, projectors $M_{H\|t}$ and $M_{F\|t}$ (and exchange rate e_t) belong to the same risk space. Hence, any traded return in that space can be priced by either $M_{H\|t}$ or $M_{F\|t}$, and the resulting return differential can be fully attributed to the exchange rate factor, or $e_t = \frac{M_{H\|t}}{M_{F\|t}}$. A prominent special case is the pure diffusion setting, in which return spaces are always identical and independent of the currency denominations. The application of Theorem 2 then establishes that the asset market view of the exchange rate determination always holds in settings of pure diffusion risks, whether financial markets are complete or incomplete. The same conclusion applies for settings of jump diffusion risks of the exchange rate, in which every jump risk is separately contracted in financial markets.[23]

On the other hand, when asset return spaces in foreign and home currencies differ, $M_{H\|t}$ and $M_{F\|t}$ belong to different spaces and Theorem 2 establishes that their wedge is not offset by the exchange rate, $\frac{M_{H\|t}}{M_{F\|t}} \neq e_t$. An example is the presence of unhedged jump risks in the exchange rate, which belong to the class of entangled risks, and hence, invalidates the asset market view of the exchange rate determination. In this regard, Brunnermeier et al. (2008) and Farhi et al. (2015) document prominent jump risks in FX markets, thus lend support for the practical relevance of the exchange rate risk entanglement.

To further illustrate the risk entanglement notion in various settings, we map the discrete market configurations of Tables 1 and 2 into their counterparts in continuous settings, in conjunction with an application of Theorem 1.

1. Disentanglement: The discrete market configuration of Table 1 corresponds to a continuous market setting subject to four uncorrelated jump types $\mathcal{J} = \{1, 2, 3, 4\}$, which map into five states,[24] and the same four traded assets $\{B_H, \frac{B_F}{e}, Y_1, Y_2\}$.[25] To be specific, state $s = 1$ corresponds to no jumps taking place, while $s = i + 1$ corresponds to a jump of type i

[23] Because these settings are a special case of complete disentanglement of exchange rate risks, Theorem 1.

[24] In continuous time, the probability that jumps of two or more types take place within the time span of dt scales as dt^2, so is almost surely zero in the mean square norm. Therefore, $S-1$ uncorrelated jump types map into S states: one state is associated each jump type taking place, and one additional state associated no jumps taking place at all.

[25] All jump types have identical arrival intensity λ to reproduce equal probabilities for all states $s \in \mathcal{S}$ in Table 1.

taking place, for $i \in \{1,2,3,4\}$. Hence, the identical exchange rate gross growth in state $s \in \{1,2,3\}$ in Table 1 maps into the absence of jump types $i \in \{1,2\}$ in the exchange rate in a continuous setting (zero jump sizes $\delta_{ei} = 0$, $i \in \{1,2\}$), i.e., only jump types $i \in \{3,4\}$ are exchange rate risks ($\delta_{ei} \neq 0$, $i \in \{3,4\}$). Following the change of basis returns (5), there exist traded portfolios $\{AD_i\}$, $i \in \{3,4\}$ (i.e., Arrow-Debreu assets), each of which loads only on a single jump type $i \in \{3,4\}$ of these exchange rate jump risks. To summarize, the market configuration in the discrete setting of Table 1 is first transformed to Arrow-Debreu assets (5), and then is mapped into the following processes in the continuous setting (see (7)), $\forall i \in \{3,4\}$,

$$\frac{dAD_{it+dt}}{AD_{it}} = \mu_i dt + \delta_{ADi}\left(d\mathcal{N}_{it} - \lambda dt\right); \qquad \frac{de_{t+dt}}{e_t} = \mu_e dt + \sum_{i \in \{3,4\}} \delta_{ei} d\mathcal{N}_{it},$$

where δ_{ADi} and δ_{ei} denote jump sizes (associated with type i) in asset AD_i's return and the exchange rate (7) respectively. Per Definition 1, the availability of these traded assets $\{AD_i\}$, $i \in \{3,4\}$ signifies the complete disentanglement of exchange rate risks in the market configuration of Table 1. Theorem 1 then confirms the incomplete asset market view of the exchange rate determination $e = \frac{M_{H\|}}{M_{F\|}}$ in that market configuration.

2. Entanglement: The discrete market configuration of Table 2 corresponds to a continuous market setting which is quite similar to the continuous setting associated with Table 1 discussed above.[26] Exchange rate jump risks are also identical: the exchange rate is exposed only to jumps of type $i \in \{3,4\}$. The key difference is in asset markets, arsing from the presence of asset Y_1' in Table 2 (as a replacement for asset Y_1 in Table 1). As a result, there do not exist traded portfolios which load only on a single exchange rate jump risk of type i ($i \in \{3,4\}$). Per Definition 1, the absence of such portfolios signifies the entanglement of exchange rate risks in the market configuration of Table 2. Theorem 1 then rules out the incomplete asset market view of the exchange rate determination for this market configuration.

[26]Specifically, it has four traded assets $\{B_H, \frac{B_F}{e}, Y_1', Y_2\}$ subject to 4 uncorrelated jump types $\mathcal{J} = \{1,2,3,4\}$. State $s = 1$ corresponds to no jumps taking place, while the remaining four states $s \in \{2,3,4,5\}$ correspond to jumps of a single type taking place.

4 Revisiting the International Correlation Puzzle

In this section, we discuss the international correlation puzzle, i.e., the observation that country-specific macroeconomic quantities modestly correlate while currencies move together much more closely (i.e., smooth exchange rates). Within the basic economic framework in which SDFs are monotone functions of macro fundamentals (e.g., consumptions), this anomaly purportedly illustrates the disconnection between quantities and prices, and thus is perplexing from the asset market view of the exchange rate determination (Brandt et al., 2006). The approach of the current literature addressing this disconnection is structural. By postulating and equipping SDFs with structural features, economic models can enrich and weaken the dynamics between SDFs and macro fundamentals. As a result, prices arising from the enriched SDFs can be dissociated from fundamental quantities. Thanks to this dissociation in structural models, SDFs are still highly correlated but observable macroeconomic quantities are not. In contrast, SDFs are modestly correlated in the risk entanglement approach, which is an alternative solution but can also complement the structural approach.

Specifically, we demonstrate numerically in this section that, while adhering to the pure market-based framework of the exchange rate, the observed puzzling disconnection between prices and quantities can be rationalized if exchange rate risks are sufficiently entangled in asset markets. Moreover, per Theorem 1, risk entanglement is the only possible pure market-based solution to the puzzle when markets are integrated, frictionless and free of arbitrage opportunities.

We first introduce a measure for the risk entanglement in FX markets (Section 4.3) before analyzing the international correlation anomaly conceptually (Section 4.1) and quantitatively through a numerical example (Section 4.2) in light of the exchange rate risk entanglement. Evidently, the rejection of Hypothesis H (Theorem 1) mitigates the puzzle because it is on the basis of the hypothesized equality (12) that the international correlation pattern is deemed puzzling in the first place.

4.1 Risk Entanglement Index

The one-to-one relationship between the presence of risk entanglement in FX markets and the deviation of the exchange rate from the ratio of SDF projectors (Theorem 1) has a useful application. In arbitrage-free, frictionless and integrated international financial markets, if the conditional identity

$e_t = \frac{M_{H\|t}}{M_{F\|t}}$ is broken, it must be solely due to the entanglement of exchange rate risks. Hence, the result of Theorem 1 motivates a measure of the degree of risk entanglement in FX markets.

Definition 2 (Risk Entanglement Index) *Given an international finance setting with arbitrage-free, frictionless and perfectly integrated financial markets, the exchange rate risk entanglement index I_{Et} represents the fraction of variation in the growth of the home SDF projector $M_{H\|t}$ that is not explained by the variation in the growth of the product $e_t M_{F\|t}$ of the exchange rate and the foreign SDF projector,*

$$I_{Et} = \frac{Var_t\left(\frac{dM_{H\|t+dt}}{M_{H\|t}} - \frac{d(e_{t+dt}M_{F\|t+dt})}{e_t M_{F\|t}}\right)}{Var_t\left(\frac{dM_{H\|t+dt}}{M_{H\|t}}\right)}. \tag{14}$$

The variances appearing in the above definition are total variances (see (56)), which include both diffusion and jump risks. A more explicit expression for the risk entanglement index (14) is given in equation (15) below. The rationale for this risk entanglement measure is as follows. We observe that by construction, the product $e_t M_{F\|t}$ consistently prices all traded asset returns in the home currency. Hence, it is a home pricing kernel from the perspective of foreign investors, who mechanically account for the exchange rate e_t by multiplying it to their (foreign) pricing kernel $M_{F\|t}$. A large value of I_{Et} means that the equality between $M_{H\|t}$ and $e_t M_{F\|t}$ is severely broken, and thus implicates a high degree of exchange rate risk entanglement in FX markets.[27] In particular, when there is no risk entanglement, I_{Et} is zero.

Several observations concerning the entanglement index are in order. First, the risk entanglement index is intrinsically a pure-price object. This is because its definition (14) involves only SDF projectors, which can be constructed uniquely from only asset return characteristics (abstracting from the specification of full SDFs and the underlying structural assumptions). However, prices and returns reflect investors' risk and time preferences (though not necessarily unequivocally). Hence, the value of index I_{Et} has important implications (in the form of bound tests) for the underlying structural economic models which aim to address the disconnection between quantities and prices.

Second, definition (14) concerns the exchange rate between specific countries, but it applies equally well in the presence of other countries as long as international markets are integrated (so that the same set of traded assets is common to investors in all countries). By intuitively invoking

[27]Similarly, one may employ a symmetric fraction of variation in the foreign SDF projector $M_{F\|t}$ that is not explained by the variation in the ratio $\frac{M_{H\|t}}{e_t}$, or alternatively, the fraction of variation in the exchange rate e_t that is not explained by the variation in the ratio $\frac{M_{H\|t}}{M_{F\|t}}$.

an OLS metaphor, I_{Et} is inversely related to R-square (of a linear projection of $e_t M_{F\|t}$ on $M_{H\|t}$). Therefore, the risk entanglement index is a proxy for the deviation between spaces of asset returns in the home and foreign currencies. The wedge between the two spaces is driven solely by the exchange rate risk entanglement (Theorem 2), and tends to be larger when risks are more entangled in FX markets.

Finally, an explicit expression of the risk entanglement index (14) for jump-diffusion settings is as follows,

$$I_{Et} = \frac{\left(\eta_{H\|} - \eta_{F\|} + \sigma_e\right)^T \left(\eta_{H\|} - \eta_{F\|} + \sigma_e\right) + \sum_k^J \lambda_k \left[\left(\delta_{F\|k} + 1\right)\left(\delta_{ek} + 1\right) - \left(\delta_{H\|k} + 1\right)\right]^2}{\eta_{H\|}^T \eta_{H\|} + \sum_k^J \lambda_k \delta_{H\|k}^2}, \quad (15)$$

where $\eta_{I\|}$ and $\delta_{H\|}$ denote the diffusion and jump size vectors of country I's SDF projector.[28] In the limit of a small exchange rate volatility (i.e., smooth exchange rate) we obtain an analytical inverse relationship between the risk entanglement index and the correlation of SDF projectors (equation (23) below). Within the pure asset-based framework, this relationship conceptually (Section 4.2) and quantitatively (Section 4.3) signifies the role of risk entanglement in rationalizing the international correlation pattern.

4.2 International Correlation Puzzle: Conceptual Analysis

We extend the analysis of Brandt et al. (2006) to incorporate new risk entanglement insights. One side of the international correlation puzzle concerns macroeconomic quantities in equilibrium settings, which in turn relate to the full (structural) country-specific SDFs $\{M_H, M_F\}$. From the arbitrage-free perspective, it is useful to decompose the full country-specific SDFs into pricing (i.e., SDF projectors) and non-pricing (i.e., orthogonal) components,

$$\frac{M_{It+dt}}{M_{It}} = 1 + \frac{dM_{It+dt}}{M_{It}} = 1 + \frac{dM_{I\|t+dt}}{M_{I\|t}} + \frac{dM_{I\perp t+dt}}{M_{I\perp t}}, \quad (16)$$

with $I \in \{H, F\}$ and,

$$E_t\left[\frac{M_{I\|t+dt}}{M_{I\|t}} \frac{Y_{It+dt}}{Y_{It}}\right] = 1, \quad E_t\left[\frac{dM_{I\perp t+dt}}{M_{I\perp t}} \frac{Y_{It+dt}}{Y_{It}}\right] = 0, \quad E_t\left[\frac{dM_{I\perp t+dt}}{M_{I\perp t}} \frac{dM_{I\|t+dt}}{M_{I\|t}}\right] = 0. \quad (17)$$

[28] See Equation (47) and Appendix A.2 for notation.

These orthogonality relationships yield a decomposition of the correlation of full SDFs, which is key to our analysis of the international correlation,

$$Corr_t\left(\frac{dM_{Ht+dt}}{M_{Ht}}, \frac{dM_{Ft+dt}}{M_{Ft}}\right) =$$

$$\left\{Cov_t\left(\frac{dM_{H\|t+dt}}{M_{H\|t}}, \frac{dM_{F\|t+dt}}{M_{F\|t}}\right) + Cov_t\left(\frac{dM_{H\|t+dt}}{M_{H\|t}}, \frac{dM_{F\perp t+dt}}{M_{F\perp t}}\right)\right.$$

$$\left. + Cov_t\left(\frac{dM_{H\perp t+dt}}{M_{H\perp t}}, \frac{dM_{F\|t+dt}}{M_{F\|t}}\right) + Cov_t\left(\frac{dM_{H\perp t+dt}}{M_{H\perp t}}, \frac{dM_{F\perp t+dt}}{M_{F\perp t}}\right)\right\} \times \tag{18}$$

$$\times \left[Var_t\left(\frac{dM_{H\|t+dt}}{M_{H\|t}}\right) + Var_t\left(\frac{dM_{H\perp t+dt}}{M_{H\perp t}}\right)\right]^{-\frac{1}{2}} \times \left[Var_t\left(\frac{dM_{F\|t+dt}}{M_{F\|t}}\right) + Var_t\left(\frac{dM_{F\perp t+dt}}{M_{F\perp t}}\right)\right]^{-\frac{1}{2}}.$$

We start with the standard model-free observation of Hansen and Jagannathan (1991) (H-J) that country-specific SDFs need to be sufficiently volatile to accommodate sizable empirical Sharpe ratios of the respective equity markets (which are of the order of 0.5 for developed economies). Because this observation relies only on no-arbitrage pricing considerations, its associated bounds apply in particular to the country-specific SDF projectors $M_{H\|}$, $M_{F\|}$ (see (57), Appendix A.2)

$$\left[\frac{1}{dt}Var_t\left(\frac{dM_{I\|t+dt}}{M_{I\|t}}\right)\right]^{\frac{1}{2}} \geq\approx 0.5, \qquad I \in \{H, F\}, \tag{19}$$

To highlight new perspectives of risk entanglement on the international correlation, we discuss the puzzle sequentially under two alternative premises (without and with risk entanglement).

Case of Completely Disentangled Exchange Rate Risks

In the traditional setting in which exchange rate risks are completely disentangled (Definition 1), Hypothesis H holds, and the exchange rate growth volatility is tightly linked to the correlation of SDF projectors (Brandt et al. (2006)). Indeed, under the premise of Hypothesis H,

$$Var_t\left(\frac{de_{t+dt}}{e_t}\right) = Var_t\left(\frac{\frac{M_{H\|t+dt}}{M_{F\|t+dt}}}{\frac{M_{H\|t}}{M_{F\|t}}} - 1\right) = Var_t\left(\frac{1 + \frac{dM_{H\|t+dt}}{M_{H\|t}}}{1 + \frac{dM_{F\|t+dt}}{M_{F\|t}}}\right)$$

$$\approx Var_t\left(\frac{dM_{H\|t+dt}}{M_{H\|t}}\right) + Var_t\left(\frac{dM_{F\|t+dt}}{M_{F\|t}}\right) - 2\rho_{HF\|}\left[Var_t\left(\frac{dM_{H\|t+dt}}{M_{H\|t}}\right)\right]^{\frac{1}{2}}\left[Var_t\left(\frac{dM_{F\|t+dt}}{M_{F\|t}}\right)\right]^{\frac{1}{2}},$$

where $\rho_{HF\|}$ is the (conditional) correlation of the SDF projector growths, and the approximation is at the same order of a log linearization.[29] Employing H-J bound (19) for both SDF projectors in the above relationship yields a further quantitative bound on the exchange rate growth variance, $\frac{1}{dt}Var_t\left(\frac{de_{t+dt}}{e_t}\right) \geq 0.5\left(1-\rho_{HF\|}\right)$, or equivalently,

$$\frac{1}{dt}Corr_t\left(\frac{dM_{H\|t+dt}}{M_{H\|t}}, \frac{dM_{F\|t+dt}}{M_{F\|t}}\right) = \rho_{HF\|} \geq 1 - 2\left[\frac{1}{dt}Var_t\left(\frac{de_{t+dt}}{e_t}\right)\right] \approx 0.98. \tag{20}$$

The last quantitative estimate is based on the smooth exchange rate growths with an empirical volatility of around 10%. When exchange rate risks are completely disentangled, spaces of asset returns in the home and foreign currencies are identical (Theorem 2). As a result, in absence of risk entanglement, the intra-country orthogonality relationships (17) generalize to cross-country counterparts.[30] To discern the perplexing implication of the international correlation pattern, let us adopt a pretense of moderating the right-hand side of (18). First, because the SDF orthogonal components' volatilities should be relatively small,[31] one cannot employ these small orthogonal volatilities to weaken the correlation of full SDFs by sizing up the denominator of (18). Second, by the same reason, while in principle a highly negative correlation between SDF orthogonal components would help to reduce the correlation of full SDFs (18), in practice this contribution is quantitatively small. Altogether, under the premise of relatively small volatilities of SDF orthogonal components (not to deepen the equity premium puzzle) the main contribution to the SDFs' correlation is that of the SDF projectors. Hence, (18) quantitatively reduces to (20),

$$\begin{cases} \frac{1}{dt}Corr_t\left(\frac{dM_{Ht+dt}}{M_{Ht}}, \frac{dM_{Ft+dt}}{M_{Ft}}\right) \approx \frac{1}{dt}Corr_t\left(\frac{dM_{H\|t+dt}}{M_{H\|t}}, \frac{dM_{F\|t+dt}}{M_{F\|t}}\right) \\ \\ \frac{1}{dt}Corr_t\left(\frac{dM_{H\|t+dt}}{M_{H\|t}}, \frac{dM_{F\|t+dt}}{M_{F\|t}}\right) \approx 0.98. \end{cases} \tag{21}$$

[29] This approximation holds as an almost-surely equality in the mean square convergence norm in the limit of infinitesimal time increment $dt \to 0$. Going beyond log linearization approximation involves retaining (co-)skewness and terms of other higher-order (mixed-)moments, which can be succinctly characterized by the relative entropy between the two SDF projectors.

[30] That is, assuming exchange rate risks are completely disentangled, we have

$$E_t\left[\frac{dM_{H\perp t+dt}}{M_{H\perp t}} \frac{dM_{F\|t+dt}}{M_{F\|t}}\right] = E_t\left[\frac{dM_{H\|t+dt}}{M_{H\|t}} \frac{dM_{F\perp t+dt}}{M_{F\perp t}}\right] = 0.$$

[31] The variance of the full SDF is the sum of the variances of the SDF projector and the remaining orthogonal component. As observed by Brandt et al. (2006), given the former being at least 0.5 (19), if the latter were sizable, the volatility of the full SDF would be even higher (thus, more perplexing and less desirable) than the level currently considered puzzlingly high in the equity premium puzzle literature.

The empirical cross-country correlations of macroeconomic quantities are typically much lower than the above near-perfect correlation. Within the basic economic framework in which full SDFs are monotone in consumptions and seen traditionally in the absence of risk entanglement, this is the gist of the international correlation puzzle: the implied SDF correlation is much higher than that between country-specific macro quantities which the SDFs purportedly represent. Going beyond the basic economic framework, additional structural features help dilute the monotone relationship between SDFs and macro fundamentals. Instead, we argue next that risk entanglement in FX markets can also mitigate the puzzle by nullifying the strong bound (21) while adhering to the basic economic structure.

Case of Entangled Exchange Rate Risks

In the presence of exchange rate risk entanglement (Definition 1), terms contributing to the correlation of the full SDFs (18) change in several notable ways.

First, because the spaces of asset returns in the home and foreign currency denomination do not coincide in the presence of entangled exchange rate risks (Theorem 2), cross-country orthogonality relationships do not hold,

$$E_t\left[\frac{dM_{H\perp t+dt}}{M_{H\perp t}}\,\frac{dM_{F\| t+dt}}{M_{F\| t}}\right]\neq 0,\qquad\qquad E_t\left[\frac{dM_{H\| t+dt}}{M_{H\| t}}\,\frac{dM_{F\perp t+dt}}{M_{F\perp t}}\right]\neq 0.\tag{22}$$

As a result, the cross terms in (18) do not vanish, and now contribute conceptually to the correlation of the full SDFs. However, these contributions remain quantitatively insignificant if one posits small volatilities of SDF orthogonal components (not to deepen the equity premium puzzle). Under such a premise, the full correlation (18) is still quantitatively dominated by the correlation of SDF projectors, and the first approximate relationship in (21) continues to hold.

Second and most crucially, entangled exchange rate risks unambiguously break the equality between the exchange rate growth and the growth of the ratio of SDF projectors (Theorem 1). As a result, bound (20), and thus, the second relationship in (21) do not necessarily hold. While projectors $M_{H\|}$ (9), $M_{F\|}$ (11) are necessarily and clearly constructs endogenous to the given asset returns (and the exchange rate e), the relationship between the ratio $\frac{M_{H\|}}{M_{F\|}}$ and the original exchange rate e can be distorted severely and far from an equality due to risk entanglement in FX markets. Intuitively, such a distortion is reflected in the intricate and flexible configuration of possibly many jump risk types embedded in few available traded assets $\left\{B_H, \frac{B_F}{e}, \{Y_n\}_{n=1}^{N-1}\right\}$. Moreover, our entangled risk approach respects the constraint imposed by the equity premium puzzle consideration (first equation in (21)), the correlation of the full SDFs (18) is substantially weakened principally because the correlation of SDF projectors is substantially weakened. To see how the latter is achieved through risk entanglement, we observe that an empirically smooth exchange rate growth implies an approximate relationship (see (60), Appendix A.2),

$$Corr_t\left(\frac{dM_{H\|t+dt}}{M_{H\|t}}, \frac{dM_{F\|t+dt}}{M_{F\|t}}\right) \approx \frac{1}{2g_e} \times \left[1 + g_e^2 - I_{Et}\right], \tag{23}$$

where $g_e \equiv E_t\left[\frac{e_{t+dt}}{e_t}\right]$ is the conditional mean of the exchange rate (gross) growth (note that $g_e \approx 1$), and I_{Et} is the exchange rate risk entanglement index given in Definition 2. We observe from (23) that a moderate correlation between SDF projectors is obtained when we are sufficiently far away from the hypothesized equality relationship (13), i.e., the risk entanglement index (14) is sufficiently high. Such a situation materializes when risks are sufficiently entangled in FX markets. We next provide a numerical illustration.

4.3 International Correlation Puzzle: Numerical Analysis

To illustrate the quantitative implications of risk entanglement, we consider a parsimonious setting of international financial markets. Our aim is to provide one of the simplest possible pure market-based models to demonstrate the potential of risk entanglement in FX markets.

Our simple numerical setting has three assets, namely the home bond the foreign bond, and one additional asset. There are two diffusion and one jump risk processes, so asset markets are incomplete. In the home currency denomination, the three assets are denoted respectively as B_H, $\frac{B_F}{e}$ and Y_1, where the exchange rate e quotes the amount of foreign currency units that buys one unit of the home currency. Asset returns in the home currency denomination and the exchange rate process are given in the setting. We configure the risk loading parameters σ_{e1}, σ_{e2}, σ_{11}, σ_{12}, δ_e, δ_1, such that the jump risk cannot be separately contracted in asset markets. Therefore, risks affecting the exchange rate are entangled. We construct the SDF projector $M_{H\|}$ using (9), and foreign asset returns and SDF projector using (11).

Table 3 summarizes the (annualized) parameters for our numerical illustration.[32] For specificity, we calibrate risky asset Y_1 to the home stock market with an expected excess return of $\mu_Y - r_H =$

[32] The risk-free interest rates in both currencies are not material for our analysis and set to 1% without loss of generality.

Table 3: Model Parameters

Jump Intensity:	$\lambda = 0.03$			
	Home Risk-free Rate		Foreign Risk-free Rate	
Bonds:	$r_H = 0.01$		$r_F = 0.01$	
	Drift	Diffusion 1	Diffusion 2	Jump
Exchange Rate:	$\mu_e = 0.06$	$\sigma_{e1} = 0$	$\sigma_{e2} = 0.075$	$\delta_e \in [-0.9, -0.2]$
asset Y_1:	$\mu_1 = 0.07$	$\sigma_{11} = 0.03$	$\sigma_{12} = -0.15$	$\delta_1 = 0$

Notes: Exogenous parameters specifying:
(1) Risk-free interest rates in home and foreign currencies,
(2) Exchange rate e: $\frac{e_{t+dt}}{e_t} = 1 + \mu_e dt + \sigma_{e1} dZ_{1t} + \sigma_{e2} dZ_{2,t} + \delta_e d\mathcal{N}_t$,
(3) Risky Asset Y_1: $\frac{Y_{1t+dt}}{Y_{1t}} = 1 + \mu_1 dt + \sigma_{11} dZ_{1t} + \sigma_{12} dZ_{2t} + \delta_1 (d\mathcal{N}_t - \lambda dt)$.

6% and a total volatility of $\sqrt{\frac{1}{dt} Var_t \left(\frac{dY_{1t+dt}}{Y_{1t}}\right)} = 15.3\%$, implying a Sharpe ratio of almost 40%. Asset Y_1 has no exposure to the jump risk, $\delta_1 = 0$. We calibrate a drift term of 6% and a volatility from a diffusion of $\sqrt{\sigma_{1,e}^2 + \sigma_{2,e}^2} = 7.5\%$ for the exchange rate growth. We let the jump size δ_e vary between -0.9 and -0.2 and set the jump intensity to 3%. These parametric values model infrequent (jump) events in FX markets. Jumps increase the exchange rate's total volatility to $\sqrt{\frac{1}{dt} Var_t \left(\frac{de_{t+dt}}{e_t}\right)} = \sqrt{\sigma_{e1}^2 + \sigma_{e2}^2 + \lambda \delta_e^2} \in [8.3\%, 17.3\%]$. Notice that the increase in the total volatility is moderate because jumps are infrequent. Hence, the exchange rate remains smooth across the entire calibrated spectrum of jump sizes.

Figure 1 reports the correlation between SDF projectors $\frac{1}{dt} Corr_t \left(\frac{dM_{H\|t+dt}}{M_{H\|t}}, \frac{dM_{F\|t+dt}}{M_{F\|t}}\right)$ (solid red line) and the exchange rate volatility $\sqrt{\frac{1}{dt} Var_t \left(\frac{de_{t+dt}}{e_t}\right)}$ (dashed black line) in the left panel, and the volatility of the home and foreign SDF projectors $\sqrt{\frac{1}{dt} Var_t \left(\frac{dM_{I\|t+dt}}{M_{I\|t}}\right)}$ (solid red line: $I = H$; dashed black line: $I = F$) in the right panel (plotted against the exchange rate jump size δ_e). The correlation between SDF projectors monotonically decreases with the size of infrequent exchange rate jumps. While the exchange rate volatility increases only slightly with the jump size, the simultaneous decrease in the correlation between SDF projectors is substantial. Moreover, SDF volatilities are above 40% at all times. Therefore, entangled jump risks in exchange rate growths are able to mitigate the correlation puzzle in the pure market-based framework: the projected SDFs are volatile and modestly correlated, whereas the exchange rate is sufficiently smooth.[33] Observe

[33] While full SDFs are not modeled in the pure market-based framework, this calibration analysis indicates that

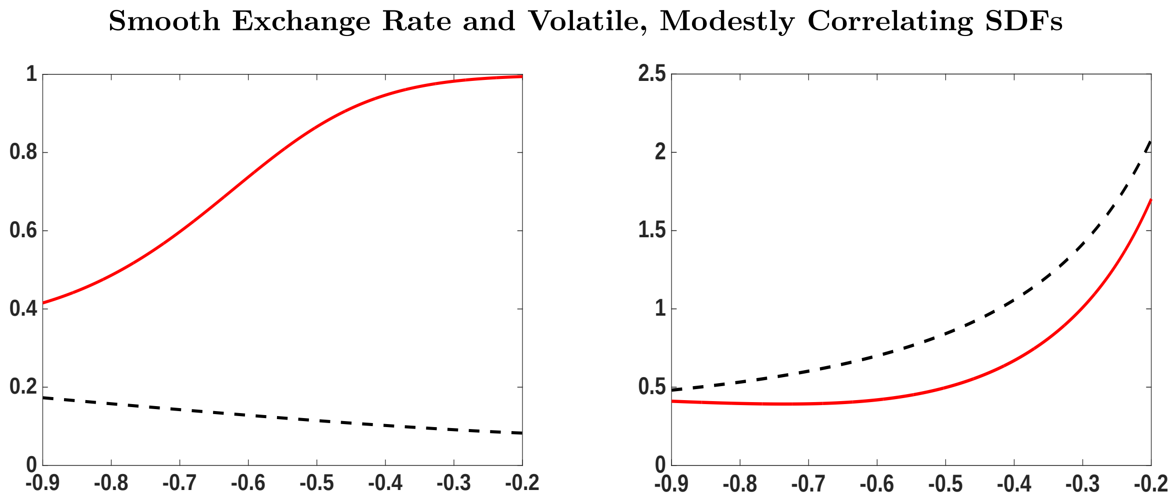

Figure 1: Left Panel: The solid red line shows the correlation between SDF projectors $\frac{1}{dt}Corr_t\left(\frac{dM_{H\|t+dt}}{M_{H\|t}}, \frac{dM_{F\|t+dt}}{M_{F\|t}}\right)$, and the dashed black line the exchange rate volatility $\sqrt{\frac{1}{dt}Var_t\left(\frac{de_{t+dt}}{e_t}\right)}$. Right Panel: The solid red line displays the volatility of the home SDF projector $\sqrt{\frac{1}{dt}Var_t\left(\frac{dM_{H\|t+dt}}{M_{H\|t}}\right)}$, and the dashed black line the volatility of the foreign SDF projector $\sqrt{\frac{1}{dt}Var_t\left(\frac{dM_{F\|t+dt}}{M_{F\|t}}\right)}$. Horizontal axis indicates the exposure of exchange rate to jump risk δ_e (7). In the underlying setting, larger jump size (more negative δ_e) regulates larger risk entanglement.

that the correlation of the SDF projectors approaches one as the jump size approaches zero (pure diffusion, and hence, completely disentangled risk limit).

Finally, we demonstrate the quantitative effect of risk entanglement on the international correlation by contrasting it with a calibration without entanglement. That is, we compare our numerical calibration above to one featuring the same SDF volatilities but the exchange rate being identified with the ratio of SDF projectors. Hence, exchange rate risks in the latter calibration are completely disentangled (Theorem 1). Specifically, consider the calibration with entangled risks and an exchange rate jump size of $\delta_e = -0.8$, exchange rate volatility of 15.8%, home and foreign SDF projector volatilities of 40% and 53%, and SDF projectors' correlation of 48.6%. The exchange rate volatility is low because in presence of entangled risks, the exchange rate is no longer equal to the ratio of SDF projectors. The comparative calibration without risk entanglement has the same home and foreign SDF projector volatilities (of 40% and 53%), and preserves equality (12) (Hypothesis H). As a result, if the correlation between the two SDF projectors was 48.6%, the exchange rate volatility would have to be equal to $\sqrt{\frac{1}{dt}Var_t\left(\frac{de_{t+dt}}{e_t}\right)} = \sqrt{0.4^2 + 0.53^2 - 2 * 0.486 * 0.4 * 0.53} = 48.5\%$,

when full SDFs are sufficiently close to their projectors, the risk entanglement approach also applies. Such a solution is impossible without risk entanglement (see Brandt et al. (2006) and Section 4.2).

which is unreasonably large and more than threefold the volatility in the calibration with entangled risks. By the same token, if the exchange rate volatility was 15.8%, the correlation between the SDF projectors would have to be $\frac{1}{dt}Corr_t\left(\frac{dM_{H\|t+dt}}{M_{H\|t}}, \frac{dM_{F\|t+dt}}{M_{F\|t}}\right) = \frac{0.4^2+0.53^2-0.158^2}{2*0.4*0.53} = 98\%$, which is unreasonably high and more than twofold the correlation in the calibration with entangled risks.

This simple numerical example demonstrates the role of exchange rate risk entanglement in rationalizing the international correlation pattern. More broadly, the modeling spirit of risk entanglement is that, by making risks more entangled in FX markets, e.g., by modeling more risk types and relatively less traded assets, one achieves larger flexibilities to calibrate observed moments in the data.

5 Conclusion

We reformulate the incomplete asset market view of the exchange rate determination in the presence of risk entanglement. Our results are as follows. For arbitrage-free, frictionless and perfectly integrated international financial markets, we show that the exchange rate is equal to the ratio of SDF projectors if and only if exchange rate risks are completely disentangled. Consequently, we demonstrate that a smooth exchange rate does not necessarily implicate a strong correlation between SDF projectors when exchange rate risks are entangled. We introduce an index to quantify the degree of risk entanglement in FX markets. When the risk entanglement index is higher, the asset market view tends to have weaker implications on the exchange rate determination. Altogether, our paper indicates that how risks are contracted in incomplete asset markets (i.e., risk entanglement) conceptually is important to rationalize the observed perplexing disparity in cross-border correlations of prices and macro fundamentals.

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Appendices

A Preliminaries

A.1 SDF Projectors in Discrete Settings

In discrete settings, the construction of SDF projector on either gross or net asset returns produces consistent and equivalent result. The construction of SDF projector in continuous settings is given in Appendix A.3. Without loss of generality, the construction is presented for the home country.

SDF projector on gross asset returns: We start with specifying $N+1$ non-redundant basis assets in international financial markets, namely the home bond, the foreign bond, and $N-1$ other assets. Because the foreign bond is a risky asset from home investors' perspective, we include it in the set of N risky assets $\{Y_n\}$, $n \in \{1, \ldots, N\}$ in the home currency denomination for ease of notation in this appendix. Accordingly, we work with the following $N+1$ asset returns in the home currency denomination,

$$\frac{B_{Ht+1}(s)}{B_{Ht}} = 1 + r_H, \qquad \frac{Y_{nt+1}(s)}{Y_{nt}} = 1 + \overline{y}_n + \tilde{y}_n(s), \qquad n \in \mathcal{N} \equiv \{1, \ldots, N\}, \ \ s \in \mathcal{S} \equiv \{1, \ldots, S\}, \tag{24}$$

where r_H is the home risk-free rate, $\overline{y}_n = E_t\left[\frac{Y_{nt+1}(s)}{Y_{nt}}\right] - 1$ the conditional net mean return, and $\tilde{y}_n(s)$ the mean-zero innovations to asset Y_n's return in the home currency. Markets are possibly incomplete ($N+1 \leq S$).

We find the pricing-consistent SDF projector $\mathcal{M}_{H\|}$ by solving for weights β's in a linear combination of all $N+1$ gross asset returns,

$$\frac{\mathcal{M}_{H\|t+1}(s)}{\mathcal{M}_{H\|t}} = \beta_H(1 + r_H) + \sum_{n}^{N} \beta_n \frac{Y_{nt+1}(s)}{Y_{nt}}, \qquad \forall s \in \mathcal{S}. \tag{25}$$

We substitute this linear representation into N Euler pricing equations $E_t\left[\frac{\mathcal{M}_{H\|t+1}}{\mathcal{M}_{H\|t}}\left(\frac{Y_{nt+1}}{Y_{nt}} - \frac{B_{Ht+1}}{B_{Ht}}\right)\right] = 0$ for N home risky assets $\{Y_n\}$, $n \in \{1, \ldots, N\}$ (24). This yields a linear system of N equations and N unknowns $\{\beta_i\}$ ($i \in \{1, \ldots, N\}$),

$$\sum_{i=1}^{N} \beta_i E_t\left[\widetilde{y}_{it+1}\widetilde{y}_{nt+1}\right] + \frac{\overline{y}_{nt} - r_H}{1 + r_H} = 0, \qquad n \in \{1, \ldots, N\}, \tag{26}$$

whose unique solution is,

$$\begin{bmatrix} \beta_1 \\ \vdots \\ \beta_N \end{bmatrix} = \frac{-1}{1+r_H} \left[\Sigma^T\right]^{-1} \begin{bmatrix} \overline{y}_{1t} - r_H \\ \vdots \\ \overline{y}_{Nt} - r_H \end{bmatrix}, \text{ where: } \Sigma \equiv \begin{bmatrix} E_t\left[\widetilde{y}_{1t+1}^2\right] & \dots & E_t\left[\widetilde{y}_{1t+1}\widetilde{y}_{Nt+1}\right] \\ \vdots & \dots & \vdots \\ E_t\left[\widetilde{y}_{Nt+1}\widetilde{y}_{1t+1}\right] & \dots & E_t\left[\widetilde{y}_{Nt+1}^2\right] \end{bmatrix}. \tag{27}$$

Finally, the remaining weight β_H is determined from N weights above and the Euler pricing equation for the home bond, $E_t\left[\frac{\mathcal{M}_{H\|t+1}}{\mathcal{M}_{H\|t}}(1+r_H)\right] = 1$,

$$\beta_H = \frac{1}{1+r_H}\left[\frac{1}{1+r_H} - \sum_{n=1}^{N} \beta_n(1+\overline{y}_{nt})\right]. \tag{28}$$

SDF projector on net asset returns: In this alternative approach, we find the SDF projector $M_{H\|}$ by solving for weights b's in a linear combination of all net returns on assets,

$$\frac{M_{H\|t+1}(s)}{M_{H\|t}} = 1 + b_H r_H + \sum_{n}^{N} b_n \left(\frac{Y_{nt+1}(s)}{Y_{nt}} - 1\right), \qquad \forall s \in \mathcal{S}. \tag{29}$$

The substitution of this projector into N Euler pricing equations $E_t\left[\frac{M_{H\|t+1}}{M_{H\|t}}\left(\frac{Y_{nt+1}}{Y_{nt}} - \frac{B_{Ht+1}}{B_{Ht}}\right)\right] = 0$ for N home risky assets specified in (24) yields a linear system of N equations and N unknowns $\{b_i\}$ $(i \in \{1, \dots, N\})$,

$$\sum_{i=1}^{N} b_i E_t\left[\widetilde{y}_{it+1}\widetilde{y}_{nt+1}\right] + \frac{\overline{y}_{nt} - r_H}{1+r_H} = 0, \qquad n \in \{1, \dots, N\}.$$

This system is identical to (26), hence its unique solution coincides with (27),

$$\begin{bmatrix} b_1 \\ \vdots \\ b_N \end{bmatrix} = \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_N \end{bmatrix} = \frac{-1}{1+r_H} \left[\Sigma^T\right]^{-1} \begin{bmatrix} \overline{y}_{1t} - r_H \\ \vdots \\ \overline{y}_{Nt} - r_H \end{bmatrix}, \tag{30}$$

with the covariance matrix Σ given in (27). The remaining weight b_0 is determined from N weights above and the Euler pricing equation for the home bond, $E_t\left[\frac{M_{H\|t+1}}{M_{H\|t}}(1+r_H)\right] = 1$,

$$b_0 = \frac{1}{r_H}\left[\frac{1}{1+r_H} - 1 - \sum_{n=1}^{N} b_n \overline{y}_{nt}\right]. \tag{31}$$

Equivalence of the two projectors: by comparing (25) with (29) (using solutions (28), (30) and (31)), we have,

$$\frac{\mathcal{M}_{H\|t+1}(s)}{\mathcal{M}_{H\|t}} = \frac{M_{H\|t+1}(s)}{M_{H\|t}}.$$

That is, *in discrete setting*, SDF projectors on gross and net asset returns are identical.

A.2 Continuous Settings: Notation and Properties

The notation for discrete settings is self-contained in Section 2. Here, we instead recapitulate the notation used in continuous settings (Sections 3-4).

Let B_I denote the risk-free bond (a.k.a., the money market account) which earns the instantaneously risk-free rate r_I when denominated in country I's currency, $\frac{B_{It+dt}}{B_{It}} - 1 = \frac{dB_{It+dt}}{B_{It}} = r_I dt$, $I \in \{H, F\}$. The n-th risky asset gross return follows a jump-diffusion process in home currency ($n \in \{1, \ldots, N\}$),

$$\begin{aligned} \frac{dY_{nt+dt}}{Y_{nt}} &= \mu_n dt + \sum_{i=1}^{d} \sigma_{ni} dZ_{it} + \sum_{i \in \mathcal{J}} \left(e^{\Delta_{ni}} - 1\right) d\mathcal{N}_{it} - \sum_{i \in \mathcal{J}} \left(e^{\Delta_{ni}} - 1\right) \lambda_i dt \\ &\equiv \mu_n dt + \sigma_n^T dZ_t + \delta_n^T d\mathcal{N}_t - \delta_n^T \lambda dt, \end{aligned} \tag{32}$$

where,

$$\sigma_n \equiv \begin{bmatrix} \sigma_{n1} \\ \vdots \\ \sigma_{nd} \end{bmatrix}, \quad Z_t \equiv \begin{bmatrix} Z_{1t} \\ \vdots \\ Z_{dt} \end{bmatrix}, \quad \delta_n = \begin{bmatrix} \delta_{n1} \\ \vdots \\ \delta_{nJ} \end{bmatrix} \equiv \begin{bmatrix} e^{\Delta_{n1}} - 1 \\ \vdots \\ e^{\Delta_{nJ}} - 1 \end{bmatrix}, \quad \mathcal{N}_t \equiv \begin{bmatrix} \mathcal{N}_{1t} \\ \vdots \\ \mathcal{N}_{Jt} \end{bmatrix},$$
$$\lambda \equiv \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_J \end{bmatrix}, \qquad \Lambda \equiv Diag(\lambda) = \begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_J \end{bmatrix}. \tag{33}$$

There are d diffusion risks and J types of jump risks (indexed by $\mathcal{J} \equiv \{1, \ldots, J\}$) in the economy. Accordingly, $d \times 1$ vector Z_t denotes a standard d-dimensional (independent) Brownian motion; $J \times 1$ vector $\mathcal{N}_t$ a standard J-dimensional (independent) Poisson counting process of corresponding arrival intensities in $J \times 1$ vector λ; $d \times 1$ vector σ_n the volatilities; and $J \times 1$ vector δ_n the net jump sizes of the return on asset Y_n. When a jump of type $i \in \mathcal{J}$ arrives, $\mathcal{N}_{it}$ increases by one, and

n-th asset's gross return increases instantly by a multiplicative factor of $e^{\Delta_{ni}}$, or,[34]

$$\frac{Y_{nt^+}}{Y_{nt}} \equiv 1 + \frac{dY_{nt^+}}{Y_{nt}} = e^{\Delta_{ni}}, \qquad \forall n \in \{1, \ldots, N\}, \;\; \forall n \in \mathcal{J}.$$

Throughout, we adopt a convention for the jump risk notation such that if the jump risk of type i does not affect asset Y_n's return, then we identically set $\Delta_{ni} \equiv 0$.

Stacking returns on all N risky assets yields,

$$\begin{bmatrix} \frac{dY_{1t}}{Y_{1t}} \\ \vdots \\ \frac{dY_{Nt}}{Y_{Nt}} \end{bmatrix} = \underbrace{\begin{bmatrix} \mu_1 \\ \vdots \\ \mu_N \end{bmatrix}}_{\mu} dt + \underbrace{\begin{bmatrix} \sigma_{11} & \cdots & \sigma_{1d} \\ \vdots & \ddots & \vdots \\ \sigma_{N1} & \cdots & \sigma_{Nd} \end{bmatrix}}_{\sigma^T} dZ_t + \underbrace{\begin{bmatrix} \delta_{11} & \cdots & \delta_{1J} \\ \vdots & \ddots & \vdots \\ \delta_{N1} & \cdots & \delta_{NJ} \end{bmatrix}}_{\delta^T} \left[d\mathcal{N}_t - \lambda dt\right]. \tag{34}$$

As an illustration of the notation above, consider a portfolio P of weights $\{\alpha_H, \{\alpha_n\}_{n=1}^N\}$ on respective assets $\{B_H, \{Y_n\}_{n=1}^N\}$ (8), (32),

$$P_t = \alpha_H B_{Ht} + \sum_{n=1}^N \alpha_n Y_{nt}, \qquad \alpha_H + \sum_{n=1}^N \alpha_n = 1.$$

The return on portfolio P is,

$$\frac{P_{t+dt}}{P_t} - 1 = \frac{dP_{t+dt}}{P_t} = \mu_P dt + \sigma_P^T dZ_t + \delta_P^T \left(d\mathcal{N}_t - \lambda dt\right), \tag{35}$$

with,

$$\mu_P = \frac{\alpha_H B_{Ht} r_H + \sum_{n=1}^N \alpha_n Y_{nt} \mu_n}{P_t}, \qquad \sigma_P = \begin{bmatrix} \sigma_{P1} \\ \vdots \\ \sigma_{Pd} \end{bmatrix} = \begin{bmatrix} \frac{\sum_{n=1}^N \alpha_n Y_{nt} \sigma_{n1}}{P_t} \\ \vdots \\ \frac{\sum_{n=1}^N \alpha_n Y_{nt} \sigma_{nd}}{P_t} \end{bmatrix},$$

$$\delta_P = \begin{bmatrix} \delta_{P1} \\ \vdots \\ \delta_{PJ} \end{bmatrix} = \begin{bmatrix} e^{\Delta_{P1}} - 1 \\ \vdots \\ e^{\Delta_{PJ}} - 1 \end{bmatrix} = \begin{bmatrix} \frac{\sum_{n=1}^N \alpha_n Y_{nt} e^{\Delta_{n1}}}{P_t} - 1 \\ \vdots \\ \frac{\sum_{n=1}^N \alpha_n Y_{nt} e^{\Delta_{nJ}}}{P_t} - 1 \end{bmatrix}.$$

Notice that the portfolio return (35) has a compensated jump-diffusion form (similar to asset returns (8)), so that μ_P is the (risk-compensated) expected return, $\mu_P = \frac{1}{dt} E_t\left[\frac{dP_{t+dt}}{P_t}\right]$. We now employ

[34] Equivalently, the return's growth rate (or log return) increases by an additional factor of Δ_{ni}. Only when jump sizes are small (which we do not assume in the current paper) $\delta_{ni} \equiv e^{\Delta_{ni}} - 1 \approx \Delta_{ni}$.

this notation in the exchange rate and return processes in the foreign currency.

Asset Returns in the Foreign Currency (10): Substituting the asset returns $\{Y_n\}$ (32) (in the home currency) and the exchange rate process e (7) and applying Ito's lemma yield the asset returns $\{Y_{Fn} \equiv eY_n\}$ in the foreign currency (10), in which,

$$\mu_{Fn} = \mu_e + \mu_n + \sigma_n^T \sigma_e + \sum_{i \in \mathcal{J}} \lambda_i \left(e^{\Delta_{ei}+\Delta_{ni}} - 1\right) - \sum_{i \in \mathcal{J}} \lambda_i \left(e^{\Delta_{ni}} - 1\right)$$

$$\sigma_{Fn} = \sigma_e + \sigma_n, \qquad \delta_{Fn} \equiv \begin{bmatrix} e^{\Delta_{Fn1}} - 1 \\ \vdots \\ e^{\Delta_{FnJ}} - 1 \end{bmatrix} = \begin{bmatrix} e^{\Delta_{e1}+\Delta_{n1}} - 1 \\ \vdots \\ e^{\Delta_{eJ}+\Delta_{nJ}} - 1 \end{bmatrix}. \tag{36}$$

Similarly, for the return on the home bond in the foreign currency eB_H, the Ito's lemma yields the return process in (10), in which,

$$\mu_{BH} = \mu_e + r_H + \sum_{i \in \mathcal{J}} \lambda_i \left(e^{\Delta_{ei}} - 1\right) = \mu_e + r_H + (\delta_e)^T \lambda,$$

$$\sigma_{BH} = \sigma_e, \qquad \delta_{BH} \equiv \begin{bmatrix} e^{\Delta_{BH1}} - 1 \\ \vdots \\ e^{\Delta_{BHJ}} - 1 \end{bmatrix} = \begin{bmatrix} e^{\Delta_{e1}} - 1 \\ \vdots \\ e^{\Delta_{eJ}} - 1 \end{bmatrix}. \tag{37}$$

Finally, for the return on the foreign bond in the home currency $\frac{B_F}{e}$, the Ito's lemma yields a return process $\frac{d(B_{Ft+dt}/e_{t+dt})}{B_{Ft}/e_t} = \mu_{BF} dt + \sigma_{BF}^T dZ_t + \delta_{BF}^T (d\mathcal{N}_t - \lambda dt)$ (8), in which,

$$\begin{gathered} \sigma_{BF} = -\sigma_e, \qquad \delta_{BFi} = e^{-\Delta_{ei}} - 1 = \tfrac{1}{1+\delta_{ei}} - 1, \qquad \forall i \in \mathcal{J}, \\ \mu_{BF} = r_F - \mu_e + \sigma_e^T \sigma_e + \sum_i^J \delta_{BFi} \lambda_i = r_F - \mu_e + \sigma_e^T \sigma_e + \delta_{BF}^T \lambda, \end{gathered} \tag{38}$$

where $\delta_{BF} = (\delta_{BF1}, \ldots, \delta_{BFJ})^T$ is the $J \times 1$ jump size vector of the foreign bond return in the home currency $\frac{B_F}{e}$.

Pricing across Currencies: Let $\widetilde{M}_H$ and $\widetilde{M}_F$ be any pricing kernels that price traded assets (8) correctly respectively in the home and foreign currency. Hence, those kernels can be the true SDFs M_H, M_F, the SDF projectors $M_{H\|}$, $M_{F\|}$, or other consistent pricing kernels. Assume these kernels follow general jump-diffusion processes,

$$\frac{\widetilde{M}_{It+dt}}{\widetilde{M}_{It}} - 1 = \frac{d\widetilde{M}_{It+dt}}{\widetilde{M}_{It}} = -r_I dt - \widetilde{\eta}_I^T dZ_t + \widetilde{\delta}_I^T \left(d\mathcal{N}_t - \lambda dt\right), \qquad I \in \{H, F\}, \tag{39}$$

where in accordance with our notation convention, the $J \times 1$ vector $\widetilde{\delta}_I$ records the jump sizes $\{\widetilde{\Delta}_i I\}$ of the growth of the pricing kernel $\widetilde{M}_I$,

$$\widetilde{\delta}_I = \begin{bmatrix} \widetilde{\delta}_{I1} \\ \vdots \\ \widetilde{\delta}_{IJ} \end{bmatrix} \equiv \begin{bmatrix} e^{\widetilde{\Delta}_{I1}} - 1 \\ \vdots \\ e^{\widetilde{\Delta}_{IJ}} - 1 \end{bmatrix}, \qquad I \in \{H, F\}.$$

The fact that $\widetilde{M}_I$, $I \in \{H, F\}$ prices all traded assets in respective currency I generates important pricing consistency constraints. We derive theses constraints explicitly before stating a key result of the no-arbitrage pricing across currencies (Proposition 1).

First, by construction, the above jump-diffusion process assures that risk-free bonds are priced correctly. Indeed, $\frac{1}{dt} E_t \left[\frac{\widetilde{M}_{It+dt}}{\widetilde{M}_{It}}\right] = -r_I$, which implies $E_t\left[\left(1 + \frac{d\widetilde{M}_{It+dt}}{\widetilde{M}_{It}}\right)(1 + r_I dt)\right] = 1$, or indeed M_I prices the risk-free bond B_I in currency I, for $I \in \{H, F\}$.[35]

Second, the home pricing of the N risky traded assets (8), $E_t\left[\frac{\widetilde{M}_{Ht+dt}}{\widetilde{M}_{Ht}} \frac{Y_{nt+dt}}{Y_{nt}}\right] = 1$, implies,

$$\mu_n - r_H = \widetilde{\eta}_H^T \sigma_n - \sum_{i \in \mathcal{J}} \lambda_i \left(e^{\widetilde{\Delta}_{Hi}} - 1\right)\left(e^{\Delta_{ni}} - 1\right) = \widetilde{\eta}_H^T \sigma_n - \widetilde{\delta}_H^T \Lambda \delta_n, \qquad \forall n \in \{1, \ldots, N\}. \tag{40}$$

Third, the foreign pricing of the n-th asset, $E_t\left[\frac{\widetilde{M}_{Ft+dt}}{\widetilde{M}_{Ft}} \frac{e_{t+dt} Y_{nt+dt}}{e_t Y_{nt}}\right] = 1$, implies for all $n \in \{1, \ldots, N\}$,

$$\mu_{Fn} - r_F = \widetilde{\eta}_F^T \sigma_{Fn} - \sum_{i \in \mathcal{J}} \lambda_i \left(e^{\widetilde{\Delta}_{Fi} + \Delta_{Fni}} - 1\right) + \sum_{i \in \mathcal{J}} \lambda_i \left(e^{\widetilde{\Delta}_{Fi}} - 1\right) + \sum_{i \in \mathcal{J}} \lambda_i \left(e^{\Delta_{Fni}} - 1\right).$$

Substituting asset return moments μ_{Fn}, σ_{Fn}, Δ_{Fni} from (36) into the above equation yields,

$$\begin{aligned} \mu_n + \mu_e - r_F &= \widetilde{\eta}_F^T(\sigma_e + \sigma_n) - \sigma_e^T \sigma_n - \textstyle\sum_{i \in \mathcal{J}} \lambda_i \left(e^{\widetilde{\Delta}_{Fi} + \Delta_{ei} + \Delta_{ni}} - 1\right) \\ &\quad + \textstyle\sum_{i \in \mathcal{J}} \lambda_i \left(e^{\widetilde{\Delta}_{Fi}} - 1\right) + \sum_{i \in \mathcal{J}} \lambda_i \left(e^{\Delta_{ni}} - 1\right). \end{aligned} \tag{41}$$

[35] After explicitly accounting for the home pricing of the home bond and the foreign pricing of the foreign bond, we are left with the following pricing equations; (i) the home pricing of the risky assets, (ii) the foreign pricing of the risky assets, among them the home bond (which indeed is a risky asset to foreign investors because of the exchange rate risks).

Fourth, the foreign pricing of the home bond, the Euler equation $E_t\left[\frac{\widetilde{M}_{Ft+dt}}{\widetilde{M}_{Ft}}\frac{e_{t+dt}B_{Ht+dt}}{e_t B_{Ht}}\right] = 1$, implies,

$$\mu_{BH} - r_F = \widetilde{\eta}_F^T \sigma_{BH} - \sum_{i\in\mathcal{J}} \lambda_i \left(e^{\widetilde{\Delta}_{Fi}+\Delta_{BHi}} - 1\right) + \sum_{i\in\mathcal{J}} \lambda_i \left(e^{\widetilde{\Delta}_{Fi}} - 1\right) + \sum_{i\in\mathcal{J}} \lambda_i \left(e^{\Delta_{BHi}} - 1\right),$$

where the notation follows from (10). Substituting μ_{BH}, σ_{BH}, Δ_{BHi} from (37) into the above equation yields,

$$\mu_e + r_H - r_F = \widetilde{\eta}_F^T \sigma_e - \sum_{i\in\mathcal{J}} \lambda_i \left(e^{\widetilde{\Delta}_{Fi}+\Delta_{ei}} - 1\right) + \sum_{i\in\mathcal{J}} \lambda_i \left(e^{\widetilde{\Delta}_{Fi}} - 1\right). \tag{42}$$

With all pricing consistency constraints having been derived, we now present an important result.

Proposition 1 *Assume arbitrage-free, frictionless and perfectly integrated international financial markets (Assumptions A1-A2). Given (i) general asset return processes (32), (ii) the general exchange rate process (7), and any general pricing kernels $\widetilde{M}_H$ and $\widetilde{M}_F$ (39) that price all traded assets in respective currencies H and F. For every traded asset Y_n we have the following pricing identity,*

$$\sigma_n^T\left(\widetilde{\eta}_H - \widetilde{\eta}_F + \sigma_e\right) + \sum_{i\in\mathcal{J}} \lambda_i \left(e^{\Delta_{ni}} - 1\right)\left(e^{\widetilde{\Delta}_{Fi}+\Delta_{ei}} - e^{\widetilde{\Delta}_{Hi}}\right) = 0, \qquad \forall n \in 1,\dots,N. \tag{43}$$

Collecting above equation for each asset Y_n, we have a system of N equations, in matrix form,

$$\underbrace{\begin{bmatrix} \sigma_{11} & \dots & \sigma_{1d} & | & \delta_{11} & \dots & \delta_{1J} \\ \vdots & \dots & \vdots & | & \vdots & \dots & \vdots \\ \sigma_{N1} & \dots & \sigma_{Nd} & | & \delta_{N1} & \dots & \delta_{NJ} \end{bmatrix}}_{\equiv A_{N\times(d+J)}} \times \begin{bmatrix} \widetilde{\eta}_{H1} - \widetilde{\eta}_{F1} + \sigma_{e1} \\ \vdots \\ \widetilde{\eta}_{Hd} - \widetilde{\eta}_{Fd} + \sigma_{ed} \\ - \; - \; - \\ \lambda_1\left(e^{\widetilde{\Delta}_{F1}+\Delta_{e1}} - e^{\widetilde{\Delta}_{H1}}\right) \\ \vdots \\ \lambda_J\left(e^{\widetilde{\Delta}_{FJ}+\Delta_{eJ}} - e^{\widetilde{\Delta}_{HJ}}\right) \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}, \tag{44}$$

where recall that δ_{ni}, $n \in \{1,\dots,N\}$, $i \in \{1,\dots,J\}$ is the jump size associated with the jump of type i in asset Y_n's return.

Proof: Observe that for each traded asset Y_n, we have a trivial identity,

$$\underbrace{\mu_n + \mu_e - r_F}_{\text{given in (41)}} = \underbrace{\mu_n - r_H}_{\text{given in (40)}} + \underbrace{\mu_e + r_H - r_F}_{\text{given in (42)}}.$$

Substituting (41), (40) and (42) into the above identity yields the desired identity (43) for each and every asset Y_n. Assembling all these equations for the set of traded assets yields (44) ■

It is important to note that in system (44), all asset characteristics are in $N \times (d+J)$ matrix A (also defined in (44)) of "coefficient", while all pricing and exchange rate characteristics are in the $(d+J) \times 1$ vectors of "unknowns". This classification and separation allows us to reformulate the fundamental issue of relating the exchange rate to country-specific pricing dynamics (SDFs) in a rigorous and quantifiable framework. Indeed, our study evolves around analyzing the consistency (i.e., solvability) of the no-arbitrage system (44).

Proposition 2 *Assume frictionless and perfectly integrated international financial markets (Assumptions A1, A2). Let e_t be the exchange rate process (7), and $\widetilde{M}_{Ht}$ (and $\widetilde{M}_{Ft}$) (39) be any pricing kernels respectively in the home (and foreign) currency. Then,*

1. *$e_t \widetilde{M}_{Ft}$ prices all traded assets consistently in the home currency. Similarly, $\frac{\widetilde{M}_{Ht}}{e_t}$ prices all traded assets consistently in the foreign currency.*
2. *If $e_t \widetilde{M}_{Ft}$ and $\widetilde{M}_{Ht}$ have matching (i.e., identical) diffusion and jump terms, then $e_t \widetilde{M}_{Ft} = \widetilde{M}_{Ht}$.*[36] *The respective result holds for the pair $\frac{\widetilde{M}_{Ht}}{e_t}$ and $\widetilde{M}_{Ft}$.*

Proof:

1. Because $\widetilde{M}_{Ft}$ prices assets in the foreign currency, we have for every traded asset Y,

$$1 = E_t \left[\frac{\widetilde{M}_{Ft+dt}}{\widetilde{M}_{Ft}} \frac{Y_{Ft+dt}}{Y_{Ft}} \right] = E_t \left[\frac{\widetilde{M}_{Ft+dt}}{\widetilde{M}_{Ft}} \frac{e_{t+dt} Y_{t+dt}}{e_t Y_t} \right] = E_t \left[\frac{e_{t+dt} \widetilde{M}_{Ft+dt}}{e_t \widetilde{M}_{Ft}} \frac{Y_{t+dt}}{Y_t} \right]. \qquad (45)$$

   Because Y is the asset's price in the home currency, the above equation is an Euler pricing equation in the home currency. This proves that $e\widetilde{M}_F$ is a consistent pricing kernel in the home currency. By similar arguments, $\frac{\widetilde{M}_H}{e}$ is a consistent pricing kernel in the foreign currency.

[36] That is, we do not need to exogenously impose the matching of drift terms (associated dt) to imply the identity $e_t \widetilde{M}_{Ft} = \widetilde{M}_{Ht}$. This matching of drift terms follows implicitly from the pricing of bonds.

2. As a special case of (45), these two pricing kernels (in the home currency) consistently price the home bond B_H, they must have identical drift terms,

$$\frac{1}{dt}E_t\left[\frac{e_{t+dt}\widetilde{M}_{Ft+dt}}{e_t\widetilde{M}_{Ft}}-1\right]=\frac{1}{dt}E_t\left[\frac{\widetilde{M}_{Ht+dt}}{\widetilde{M}_{Ht}}-1\right]=-r_H.$$

Now if we assume that $e_t\widetilde{M}_{Ft}$ and $\widetilde{M}_{Ht}$ also have matching diffusion terms (associated with dZ_t) and jump terms (associated with $d\mathcal{N}_t$), then these two kernels are matched in all terms and thus are identical (i.e., $e_t\widetilde{M}_{Ft}=\widetilde{M}_{Ht}$) almost surely under standard regularity conditions.[37] ■

A specific application of Proposition 2, in which pricing kernels $\widetilde{M}_{Ht}$, $\widetilde{M}_{Ft}$ respectively are SDF projectors $M_{H\|t}$, $M_{F\|t}$, is essential in deriving the main Theorem 1 (see Proposition 5 below).

A.3 SDF Projectors in Continuous Settings

In the difference with discrete settings, a consistent construction of SDF projector in continuous settings is necessarily on net asset returns.

(A) Uniqueness of the SDF projector:

We start by looking for the home SDF projector linear in net asset returns,

$$\frac{M_{H\|t+dt}}{M_{H\|t}}=1+\beta_H r_H dt+\sum_{n}^{N}\beta_n\left(\frac{Y_{nt+dt}}{Y_{nt}}-1\right), \tag{46}$$

which mirrors the projector (29) in discrete setting,[38] and r_H is the home risk-free rate. Next, substituting asset returns (8) in the home currency into the representation (46) and applying Ito's lemma, we can rewrite the home SDF projector growth as a standard stochastic process,

$$\begin{aligned}\frac{dM_{H\|t+dt}}{M_{H\|t}} &= -r_H dt-\eta_{H\|}^T dZ_t+\textstyle\sum_{i\in\mathcal{J}}\left(e^{\Delta_{H\|i}}-1\right)(d\mathcal{N}_{it}-\lambda_i dt)\\ &= -r_H dt-\eta_{H\|}^T dZ_t+\delta_{H\|}^T(d\mathcal{N}_t-\lambda dt),\end{aligned} \tag{47}$$

[37] These standard regularity conditions assure the uniqueness of the solution to the stochastic differential equation (39).

[38] (46) is also the representation (9), in which the foreign bond is explicitly a risky asset to home investors.

and obtain home prices of diffusion risks ($\eta_{H\|}$) and of jump risks ($\delta_{H\|}$) by respectively matching diffusion (dZ_t) and jump ($d\mathcal{N}_t$) terms. Specifically,

$$\eta_{H\|} = -\sum_{n=1}^{N} \beta_n \sigma_n = -\sigma\beta, \tag{48}$$

where σ is the $d \times N$ asset return volatility matrix (34), and $\beta \equiv (\beta_1, \ldots, \beta_N)^T$ the $N \times 1$ coefficient vector determining the home SDF projector (47), and

$$e^{\Delta_{H\|i}} - 1 = \sum_{n=1}^{N} \beta_n \left(e^{\Delta_{ni}} - 1\right), \quad \text{or in matrix notation,} \quad \delta_{H\|} = \delta^T \beta, \tag{49}$$

where δ is the $J \times N$ asset return jump size matrix (34), $\delta_{H\|} \equiv \left(e^{\Delta_{H\|1}} - 1, \ldots, e^{\Delta_{H\|J}} - 1\right)^T$ the $J \times 1$ jump size vector of the home SDF projector. Finally, the matching of drift terms dt yields,

$$-r_H - \sum_{i \in \mathcal{J}} \lambda_i \left(e^{\Delta_{H\|i}} - 1\right) = \beta_H r_H + \sum_{n=1}^{N} \beta_n \mu_n - \sum_{n=1}^{N} \beta_n \left[\sum_{i \in \mathcal{J}} \lambda_i \left(e^{\Delta_{ni}} - 1\right)\right]. \tag{50}$$

Intuitively, the requirement that the linear representation (9) of SDF projector growth prices correctly the $N+1$ basis assets $\left\{B_H, \{Y_n\}_{n=1}^N\right\}$ in the home currency generates a linear system of $N+1$ equations and $N+1$ unknowns $\left\{\beta_H, \{\beta_n\}_{n=1}^N\right\}$. As a result, there exists a unique SDF projector solution $M_{H\|t}$. Indeed, the Euler pricing equation of asset Y_n, $E_t\left[\left(1 + \frac{dM_{H\|t+dt}}{M_{H\|t}}\right)\left(1 + \frac{dY_{nt+dt}}{Y_{nt}}\right)\right] = 1$, implies the return premium on asset Y_n, $\forall n \in \{1, \ldots, N\}$, (which is a special case of (40), for which the consistent pricing kernel $\widetilde{M}_H$ is chosen to be the SDF projector $M_{H\|}$),

$$\mu_n - r_H = \eta_{H\|}^T \sigma_n - \sum_{i \in \mathcal{J}} \lambda_i \left(e^{\Delta_{H\|i}} - 1\right)\left(e^{\Delta_{ni}} - 1\right) = \eta_{H\|}^T \sigma_n - \delta_{H\|}^T \Lambda \delta_n. \tag{51}$$

where in the last expression we have used notations in (33). Substituting $\eta_{H\|}$ and $\delta_{H\|}$ from (48), (49) into the above equation gives one equation (per a priced asset Y_n) for the weights $\{\alpha_{Hn}\}$,

$$\mu_n - r_H = -\sum_{k}^{N} \beta_k \sigma_k^T \sigma_n - \sum_{k}^{N} \beta_k (\delta_k)^T \Lambda \delta_n = -\left[\sigma_n^T \sigma + (\delta_n)^T \Lambda \delta\right] \beta, \quad \forall n \in \{1, \ldots, N\}.$$

Now stacking all N equations ($n \in \{1, \ldots, N\}$) we obtain a system of equations pricing N home risky assets,

$$\begin{bmatrix} \mu_1 - r_H \\ \vdots \\ \mu_N - r_H \end{bmatrix} = -\left(\sigma^T \sigma + \delta^T \Lambda \delta\right) \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_N \end{bmatrix}, \tag{52}$$

where volatility matrix σ and jump size matrix δ of asset returns are as in (34). The above linear system yields a unique solution for the weight vector β of the home projector,

$$\beta \equiv \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_N \end{bmatrix} = -\left(\sigma^T \sigma + \delta^T \Lambda \delta\right)^{-1} \begin{bmatrix} \mu_1 - r_H \\ \vdots \\ \mu_N - r_H \end{bmatrix} \tag{53}$$

The remaining weight β_H (associated with the home bond) is determined from the requirement that $M_{H\|}$ prices the home bond. That is, by virtue of (50), $\beta_H = -1 - \frac{1}{r_H} \sum_{n=1}^{N} \beta_n \mu_n$, in which weights β have been obtained in (53). These results confirm the unique linear representation for the home SDF projector (9).

Having solved for the unique weights $\{\beta_H, \{\beta_n\}_{n=1}^N\}$, we now obtain explicit expressions for the home prices of diffusion and jump risks by substituting these weights respectively into (48) and (49),

$$\eta_{H\|} = \sigma \left(\sigma^T \sigma + \delta^T \Lambda \delta\right)^{-1} \overline{\mu}, \qquad \delta_{H\|} \equiv \begin{bmatrix} e^{\Delta_{H\|1}} - 1 \\ \vdots \\ e^{\Delta_{H\|J}} - 1 \end{bmatrix} = -\delta \left(\sigma^T \sigma + \delta^T \Lambda \delta\right)^{-1} \overline{\mu},$$

where asset return diffusion matrix σ and jump size matrix δ are given in (34).

(B) SDF projector as the minimum-variance pricing kernel:

The derivation of this result is standard and is reproduced below for the sake of completeness. Let $\widetilde{M}_H$ and $M_{H\|}$ respectively be any consistent pricing kernel (that prices assets consistently in the home currency) and the unique home SDF projector.[39] We define,

$$\frac{d\widetilde{M}_{H\perp t+dt}}{\widetilde{M}_{H\perp t}} \equiv \frac{d\widetilde{M}_{Ht+dt}}{\widetilde{M}_{Ht}} - \frac{dM_{H\|t+dt}}{M_{H\|t}}.$$

[39]Because the projector $M_{H\|}$ is unique, we simply do not associate a "hat" notation with it.

Because both $\widetilde{M}_H$ and $M_{H\|}$ price asset returns in the home currency consistently, the risk premium on a traded asset can be determined by either kernels. We have,

$$-E_t\left[\frac{d\widetilde{M}_{Ht+dt}}{\widetilde{M}_{Ht}}\frac{dY_{nt+dt}}{Y_{nt}}\right]=\mu_n-r_H=-E_t\left[\frac{dM_{H\|t+dt}}{M_{H\|t}}\frac{dY_{nt+dt}}{Y_{nt}}\right]\Rightarrow E_t\left[\frac{d\widetilde{M}_{H\perp t+dt}}{\widetilde{M}_{H\perp t}}\frac{dY_{nt+dt}}{Y_{nt}}\right]=0,$$

for all traded assets Y_n. Next, because the projector $M_{H\|}$ is linear in these asset returns (9), the last equality implies the following orthogonality,

$$E_t\left[\frac{d\widetilde{M}_{H\perp t+dt}}{\widetilde{M}_{H\perp t}}\frac{dM_{H\|t+dt}}{M_{H\|t}}\right]=0.$$

From this orthogonality follows an inequality among the total variances,

$$Var_t\left(\frac{d\widetilde{M}_{Ht+dt}}{\widetilde{M}_{Ht}}\right)=Var_t\left(\frac{dM_{H\|t+dt}}{M_{H\|t}}\right)+Var_t\left(\frac{d\widetilde{M}_{H\perp t+dt}}{\widetilde{M}_{H\perp t}}\right)\geq Var_t\left(\frac{dM_{H\|t+dt}}{M_{H\|t}}\right),$$

which completes the derivation of the projector $M_{H\|}$ (9) and its uniqueness ■

We remark that while the projection (9) and its derivation refer explicitly to the home quantities, these results arise in arbitrage-free and frictionless markets. Therefore they hold generically for any country. Furthermore, both properties (uniqueness and minimum-variance) of the SDF projector hold in general jump-diffusion settings, regardless of whether asset market risks are entangled or completely disentangled.

It is important to observe that, in continuous settings, the projection construction above necessarily concerns the net growth of the SDF projected on the net growth of asset returns. This is because, in contrast with discrete settings, the projection of the gross growth of the SDF on the gross asset returns in continuous time imposes an extra non-redundant constraint on coefficients β's in matching the free terms (beyond the matching constraints on respectively the drift (dt), diffusion (dZ_t) and jump ($d\mathcal{N}_t$) terms). As a result, in continuous settings, the extra constraint renders such a projection of SDF gross growth either infeasible or (if infeasible) pricing-inconsistent in either single-country or international (many-country) frameworks.[40]

[40]The point is, while SDF projectors are pricing kernels linear in asset returns, they are not proper portfolio returns. Therefore, the matching of the free term can be done simply in two separate steps: (i) matching SDF net growth $\frac{dM_{H\|t+dt}}{M_{H\|t}}$ with a linear combination of net asset returns as in (9), and then (ii) adding the trivial free term of 1 to both sides of (9). On the other hand, the matching of the SDF gross growth $\frac{M_{H\|t+dt}}{M_{H\|t}}$ with a linear combination of gross asset returns in a single step, $\frac{M_{H\|t+dt}}{M_{H\|t}}\equiv\beta_0(1+r_Hdt)+\sum_{n=1}^N\beta_n\frac{Y_{nt+dt}}{Y_{nt}}$, is both inconsistent and unnecessary. See (Maurer and Tran, 2016) for further details.

Accordingly, the following remark summarizes the simplified notation employed in the paper.

Remark 1 *Throughout the current paper, an equality between two stochastic processes (e.g., SDF projectors or asset returns) A_t, B_t always means the matching of the stochastic growth rates of these two processes,*

$$\frac{dA_t}{A_t} = \frac{dB_t}{B_t} \quad \longleftrightarrow \quad \textit{simplified notation: } A_t = B_t,$$

$$\textit{where:} \quad \frac{dA_t}{A_t} \equiv \frac{A_{t+dt}}{A_t} - 1, \quad \frac{dB_t}{B_t} \equiv \frac{B_{t+dt}}{B_t} - 1.$$

The above equality matching is a convergence in mean square (i.e., norm $\mathcal{L}^2$).

Hansen-Jagannathan Bounds for Jump-Diffusion Settings: For specificity, assume that country I's gross market (cum-dividend) returns follows jump-diffusion process,

$$R_{It+dt} \equiv \frac{S_{It+dt} + D_{It}dt}{S_{It}} = 1 + \mu_{SI}dt + \sigma_{SI}^T dZ_t + \sum_k^J \left(e^{\Delta_{SIk}} - 1\right)(d\mathcal{N}_{kt} - \lambda_k dt), \quad I \in \{H, F\}. \tag{54}$$

The Euler pricing equation $E_t\left[\frac{M_{I\|t+dt}}{M_{I\|t}} R_{It+dt}\right] = 1$ for country I's equity market return yields the respective equity premium,

$$\mu_{SI} - r_I = -\frac{1}{dt} Cov_t\left(\frac{dM_{I\|t+dt}}{M_{I\|t}}, R_{It+dt}\right) = -\frac{\rho_{I,SI}}{dt}\left[Var_t\left(\frac{dM_{I\|t+dt}}{M_{I\|t}}\right)\right]^{\frac{1}{2}} \left[Var_t\left(R_{It+dt}\right)\right]^{\frac{1}{2}}. \tag{55}$$

In the above expression, variances are total (i.e., consisting of both jump and diffusion risks),

$$\frac{1}{dt} Var_t\left(\frac{dM_{I\|t+dt}}{M_{I\|t}}\right) = \eta_{I\|}^T \eta_{I\|} + \lambda_k \left(e^{\Delta_{I\|k}} - 1\right)^2, \qquad \frac{1}{dt} Var_t\left(R_{It+dt}\right) = \sigma_{SI}^T \sigma_{SI} + \lambda_k \left(e^{\Delta_{SIk}} - 1\right)^2, \tag{56}$$

and $\rho_{I,SI}$ denotes the correlation between the SDF projector and the respective equity market return,

$$\rho_{I,SI} \equiv Corr_t\left(\frac{dM_{I\|t+dt}}{M_{I\|t}}, R_{It+dt}\right) = \frac{\eta_{I\|}^T \sigma_{SI} + \sum_k^J \lambda_k \left(e^{\Delta_{I\|k}} - 1\right)\left(e^{\Delta_{SIk}} - 1\right)}{\sqrt{\eta_{I\|}^T \eta_{I\|} + \sum_k^J \lambda_k \left(e^{\Delta_{I\|k}} - 1\right)^2} \sqrt{\sigma_{SI}^T \sigma_{SI} + \sum_k^J \lambda_k \left(e^{\Delta_{SIk}} - 1\right)^2}}.$$

An application of the Cauchy-Schwarz inequality, $(\sum_n U_n V_n)^2 \le (\sum_n U_n^2)(\sum_n V_n^2)$, for the following $(d+J)$-vectors U, V,

$$U \equiv \left\{\eta_{I\|i}, \sqrt{\lambda_k}\left(e^{\Delta_{I\|k}}-1\right)\right\}_{i\in\{1,\ldots,d\},k\in\{1,\ldots,J\}} \qquad V \equiv \left\{\sigma_{SIi}, \sqrt{\lambda_k}\left(e^{\Delta_{SIk}}-1\right)\right\}_{i\in\{1,\ldots,d\},k\in\{1,\ldots,J\}} \tag{57}$$

then assures a proper bound for the correlation, $|\rho_{I,SI}| \le 1$. Substituting this bound into (55) in turn gives rise to the H-J lower bound for the SDF projector's volatility (19).

Deriving Equation (23): We start with the conditional variance,

$$Var_t\left(\frac{M_{H\|t+dt}}{M_{H\|t}} - \frac{e_{t+dt}}{e_t}\frac{M_{F\|t+dt}}{M_{F\|t}}\right) \tag{58}$$

$$= Var_t\left(\frac{M_{H\|t+dt}}{M_{H\|t}}\right) + Var_t\left(\frac{e_{t+dt}}{e_t}\frac{M_{F\|t+dt}}{M_{F\|t}}\right) - 2Cov_t\left(\frac{M_{H\|t+dt}}{M_{H\|t}}, \frac{e_{t+dt}}{e_t}\frac{M_{F\|t+dt}}{M_{F\|t}}\right).$$

Now we employ two well-known (empirical and implied) features that (i) the exchange rate growth is smooth (with annual volatility around 10%), (ii) SDF projector growths are volatile (with annual volatilities around 50% as implied by the Hansen-Jagannathan bound to accommodate equity premia). Adopting these features, stochastic movements in the SDF projector growths are the main driver of the above variance. Accordingly, in the first-order approximation, we substitute the (gross) exchange rate growth $\frac{e_{t+dt}}{e_t}$ by its conditional expectation $g_e \equiv E_t\left[\frac{e_{t+dt}}{e_t}\right]$ in (58), which now becomes,

$$Var_t\left(\frac{M_{H\|t+dt}}{M_{H\|t}} - \frac{e_{t+dt}}{e_t}\frac{M_{F\|t+dt}}{M_{F\|t}}\right) \tag{59}$$

$$\approx Var_t\left(\frac{M_{H\|t+dt}}{M_{H\|t}}\right) + g_e^2 Var_t\left(\frac{M_{F\|t+dt}}{M_{F\|t}}\right) - 2g_e Cov_t\left(\frac{M_{H\|t+dt}}{M_{H\|t}}, \frac{M_{F\|t+dt}}{M_{F\|t}}\right).$$

In a symmetric and reasonable configuration in which SDF projector growths have similar volatilities, $Var_t\left(\frac{M_{H\|t+dt}}{M_{H\|t}}\right) \approx Var_t\left(\frac{M_{F\|t+dt}}{M_{F\|t}}\right)$ the above expression simplifies further to,

$$\frac{Var_t\left(\frac{M_{H\|t+dt}}{M_{H\|t}} - \frac{e_{t+dt}}{e_t}\frac{M_{F\|t+dt}}{M_{F\|t}}\right)}{Var_t\left(\frac{M_{H\|t+dt}}{M_{H\|t}}\right)} \approx \left[1 + g_e^2 - 2g_e Corr_t\left(\frac{M_{H\|t+dt}}{M_{H\|t}}, \frac{M_{F\|t+dt}}{M_{F\|t}}\right)\right], \tag{60}$$

The last equation is equivalent to (23). Also observe that because the exchange rate is smooth, its mean gross growth $g_e \approx 1$ ■

B Proofs of Main Theorems

In this Appendix we derive Theorems 1 and 2. In particular, the proof of Theorem 1 offers valuable insights into the nature of the risk entanglement concept, and it is instructive to be presented in key steps. In order, Section B.1 establishes the sufficiency and Section B.2 the necessity of relationship (13). Section B.3 derives Theorem 2 and also reflects on the asset market view of the exchange rate determination from technical aspects of risk entanglement.

B.1 Theorem 1: The Sufficient Condition

This section demonstrates that: assuming A1 and A2, if every risk impacting the exchange rate e_t can be singly traded in financial markets, then e_t is the ratio of SDF projectors,

$$\text{Exchange rate risks are completely disentangled} \quad \Longrightarrow \quad e_t = \frac{M_{H\|t}}{M_{F\|t}}, \quad \forall t \in [0, \infty) \tag{61}$$

Because the projectors (9) are constructed on the net growth quantities, the above equality rigorously presents, $\frac{de_{t+dt}}{e_t} = d\left(\frac{M_{H\|t+dt}}{M_{F\|t+dt}}\right) / \frac{M_{H\|t}}{M_{F\|t}}$ (see Remark 1, Appendix A.2 for notation).

Plan of Attack: Starting from the assumed complete disentanglement of exchange rate risks, the derivation of this sufficient condition for the Hypothesis H proceeds as follows.

S1. First, completely disentangled exchange rate risks implies that the entire asset space $\mathcal{R}$ can be decomposed into two mutually orthogonal (i.e., uncorrelated) subspaces; a set $\mathcal{R}_e$ of all assets sensitive to exchange rate risks, and a set $\overline{\mathcal{R}}_e$ of all assets neutral to exchange rate risks.

S2. Second, the components of the home and foreign SDF projectors concerning assets neutral to exchange rate risks are identical,[41] $M_{H\|}\Big|_{\overline{\mathcal{R}}_e} = M_{F\|}\Big|_{\overline{\mathcal{R}}_e}$.

S3. Third, the components of the home and foreign SDF projectors concerning assets sensitive to exchange rate risks satisfy, $M_{H\|}\Big|_{\mathcal{R}_e} = e\, M_{F\|}\Big|_{\mathcal{R}_e}$.

Taken altogether, these results prove the sufficient condition (61). We now carry out this attack plan in detail.

[41] Recall from equations (9) and (11) that SDF projectors are linear in asset returns. Therefore, each component of SDF projectors simply is a return on some asset.

Step S1: Orthogonalizing the asset risk space

Recall that $\mathcal{R}$ (defined above (8)) is the set of all asset returns in the home currency.[42] From the home currency denomination perspective, let $\mathcal{R}_e$ be the set of assets that load on exchange rate risks together with the home bond B_H, and $\overline{\mathcal{R}}_e$ the complementary set of all other assets, $\overline{\mathcal{R}}_e = \mathcal{R} \setminus \mathcal{R}_e$.

Because exchange rate risks are completely disentangled, by definition, there exists a basis for $\mathcal{R}_e$ in which each asset loads only on a single exchange rate risk. Furthermore, the decomposition $\mathcal{R} = \mathcal{R}_e \cup \overline{\mathcal{R}}_e$ has the following orthogonality and currency-neutrality properties.

Proposition 3 (Orthogonal Decomposition) *When the exchange rate risks are completely disentangled, it is possible to construct a decomposition $\mathcal{R} = \mathcal{R}_e \cup \overline{\mathcal{R}}_e$ such that no assets in $\overline{\mathcal{R}}_e$ load on exchange rate risks. Consequently, the asset returns in the home currency $\mathcal{R}$ and in foreign currency $\mathcal{R}_F$ span an identical risk space,*

$$\mathcal{R} = \mathcal{R}_e \cup \overline{\mathcal{R}}_e = \mathcal{R}_F, \qquad \mathcal{R}_e \cap \overline{\mathcal{R}}_e = \emptyset. \tag{62}$$

Proof: Consider a generic asset $\overline{Y}_i \in \overline{\mathcal{R}}_e$ that loads on exchange rate risk i of either diffusion or jump nature.[43] Let $Y_i \in \mathcal{R}_e$ be the asset that loads singly on this exchange rate risk i. Such Y_i exists because the set $\mathcal{R}_e$ is completely disentangled. By linearly combining $\overline{Y}_i$ with $Y_i \in \mathcal{R}_e$, we can form a new asset (portfolio) that does not load on risk i (see the discussion below (35)). By repeating this process, we can remove all exchange rate risks from asset $\overline{Y}_i$. This procedure then produces an asset basis for $\overline{\mathcal{R}}_e$ in which all assets are neutral to exchange rate risks.

Next, the entire asset return space in the foreign currency is captured by $\mathcal{R}_F = e\mathcal{R}_H = (e\mathcal{R}_e) \cup \left(e\overline{\mathcal{R}}_e\right)$. On one hand, because $\mathcal{R}_e$ is already completely disentangled, multiplying it by exchange rate process e (which obviously carries only exchange rate risks) does not add new risk dynamics to it. As a result we have the identity $e\mathcal{R}_e \equiv \mathcal{R}_e$. On the other hand, because the risks carried by $\overline{\mathcal{R}}_e$ are orthogonal to the exchange rate risks carried by e, we have $e\overline{\mathcal{R}}_e \equiv e \cup \overline{\mathcal{R}}_e$. Combining these two results yields,

$$\mathcal{R}_F = \mathcal{R}_e \cup \left(e \cup \overline{\mathcal{R}}_e\right) = (\mathcal{R}_e \cup e) \cup \overline{\mathcal{R}}_e = \mathcal{R}_e \cup \overline{\mathcal{R}}_e = \mathcal{R}, \tag{63}$$

[42] Formally, $\mathcal{R}$ is the set $\{B_H, \{Y_n\}_{n=1^N}\}$ when expressed symbolically in the home currency, and the set $\{B_F, \{Y_{Fn}\}_{n=1^N}\} \equiv \{B_F, \{eY_n\}_{n=1^N}\}$ when expressed symbolically in the foreign currency.

[43] Asset $\overline{Y}_i$ may also load on other (exchange rate and non exchange rate) risks as well.

where the third equality arises because all exchange rate risks carried by e are already in $\mathcal{R}_e$. The last identity completes the proof of Proposition 3 ■

Step S2: SDFs Projected on Non Exchange Rate Risk Space

Recall from either representation (9) that the unique home SDF projector is linear in asset returns in the home currency. When the asset return space is decomposed into two subspaces (as in step S1 above) in the home currency, we can also partition the home SDF projector accordingly into two groups of terms,

$$\frac{dM_{H\|t+dt}}{M_{H\|t}} = \sum_{n\in\mathcal{R}_e} \beta_n \frac{dY_{nt+dt}}{Y_{nt}} + \sum_{n\in\overline{\mathcal{R}}_e} \beta_n \frac{dY_{nt+dt}}{Y_{nt}} \equiv \left.\frac{dM_{H\|t+dt}}{M_{H\|t}}\right|_{\mathcal{R}_e} + \left.\frac{dM_{H\|t+dt}}{M_{H\|t}}\right|_{\overline{\mathcal{R}}_e}. \tag{64}$$

Similarly, for the foreign SDF projector (11), using fact (63) that the asset return space is currency-neutral yields,

$$\frac{dM_{F\|t+dt}}{M_{F\|t}} = \sum_{n\in\mathcal{R}_e} \widehat{\beta}_n \frac{dY_{Fnt+dt}}{Y_{Fnt}} + \sum_{n\in\overline{\mathcal{R}}_e} \widehat{\beta}_n \frac{dY_{Fnt+dt}}{Y_{Fnt}} \equiv \left.\frac{dM_{F\|t+dt}}{M_{F\|t}}\right|_{\mathcal{R}_e} + \left.\frac{dM_{F\|t+dt}}{M_{F\|t}}\right|_{\overline{\mathcal{R}}_e}. \tag{65}$$

Continued with the orthogonal decomposition and currency-neutrality (62) of the asset return spaces, we have a further intuitive pricing result.

Proposition 4 (Non Exchange Rate Risk Pricing) *When the exchange rate risks are completely disentangled, the non exchange rate risks earn identical compensated returns in either currencies. Consequently, SDFs projected on the non exchange rate risk space are identical,*

$$\left.\frac{dM_{H\|t+dt}}{M_{H\|t}}\right|_{\overline{\mathcal{R}}_e} = \left.\frac{dM_{F\|t+dt}}{M_{F\|t}}\right|_{\overline{\mathcal{R}}_e}. \tag{66}$$

Proof: Consider a generic asset $\overline{Y}_e \in \overline{\mathcal{R}}_e$. Observe that $\overline{Y}_e$ loads only on non-exchange rate risks because the set $\overline{\mathcal{R}}_e$ is orthogonal to $\mathcal{R}_e$ (Proposition 3).[44] Let $\overline{\mu}_{Ye}$ be the expected return of this asset in home currency, or $\overline{\mu}_{Ye} = \frac{1}{dt}E_t\left[\frac{d\overline{Y}_{et+dt}}{\overline{Y}_{et}}\right]$ (see (8)). The expected return of the same asset in foreign currency then is (see (10)),

$$\overline{\mu}_{FYe} = \frac{1}{dt}E_t\left[\frac{d(e_{t+dt}\overline{Y}_{et+dt})}{e_t\overline{Y}_{et}}\right] = \overline{\mu}_{Ye} + \mu_e, \tag{67}$$

[44] Asset $\overline{Y}_e$ may also load on multiple non exchange rate risks because the set $\overline{\mathcal{R}}_e$ is not necessarily completely disentangled.

where μ_e is the drift of the exchange rate (7). In the last equality we have used Ito's lemma and the fact that the non exchange rate risks carried by asset $\overline{Y}_e$ is orthogonal to the exchange rate risks carried by e.

The application of the Euler pricing equation in the home currency, $E_t\left[\frac{M_{H\|t+dt}}{M_{H\|t}}\frac{\overline{Y}_{et+dt}}{\overline{Y}_{et}}\right]=1$, yields the risk premium on asset $\overline{Y}_e$,

$$\overline{\mu}_{Ye}-r_H=-Cov_t\left(\left.\frac{dM_{H\|t+dt}}{M_{H\|t}}\right|_{\overline{\mathcal{R}}_e},\frac{d\overline{Y}_{et+dt}}{\overline{Y}_{et}}\right),\tag{68}$$

where we have used the representation (64) and the orthogonality between exchange rate risks in $\mathcal{R}_e$ and non exchange rate risks in $\overline{\mathcal{R}}_e$. Similarly, using (65) and the same orthogonality, the Euler pricing equation $E_t\left[\frac{M_{F\|t+dt}}{M_{F\|t}}\frac{e_{t+dt}\overline{Y}_{et+dt}}{e_t\overline{Y}_{et}}\right]=1$ in the foreign currency yields the foreign risk premium on the same asset $\overline{Y}_e$,

$$\overline{\mu}_{FYe}-r_F=-Cov_t\left(\left.\frac{dM_{F\|t+dt}}{M_{F\|t}}\right|_{\overline{\mathcal{R}}_e},\frac{d\overline{Y}_{et+dt}}{\overline{Y}_{et}}\right)-Cov_t\left(\left.\frac{dM_{F\|t+dt}}{M_{F\|t}}\right|_{\mathcal{R}_e},\frac{de_{t+dt}}{e_t}\right).\tag{69}$$

A case similar to the above premium is the foreign pricing of the home bond (8), $E_t\left[\frac{M_{F\|t+dt}}{M_{F\|t}}\frac{e_{t+dt}B_{Ht+dt}}{e_tB_{Ht}}\right]=1$, which gives rise to an identity, $\mu_e+r_H-r_F=-Cov_t\left(\left.\frac{dM_{F\|t+dt}}{M_{F\|t}}\right|_{\mathcal{R}_e},\frac{de_{t+dt}}{e_t}\right)$. Substituting this identity into premium (69), then combining it with (67) and (69) obtains,

$$Cov_t\left(\left.\frac{dM_{H\|t+dt}}{M_{H\|t}}\right|_{\overline{\mathcal{R}}_e},\frac{d\overline{Y}_{et+dt}}{\overline{Y}_{et}}\right)=Cov_t\left(\left.\frac{dM_{F\|t+dt}}{M_{F\|t}}\right|_{\overline{\mathcal{R}}_e},\frac{d\overline{Y}_{et+dt}}{\overline{Y}_{et}}\right),\qquad\forall\,\overline{Y}_e\in\overline{\mathcal{R}}_e.\tag{70}$$

This identity shows that every non exchange rate risk is priced identically by either $\left.\frac{dM_{H\|t+dt}}{M_{H\|t}}\right|_{\overline{\mathcal{R}}_e}$ or $\left.\frac{dM_{F\|t+dt}}{M_{F\|t}}\right|_{\overline{\mathcal{R}}_e}$, and therefore, also priced identically by the full projectors $\frac{dM_{H\|t+dt}}{M_{H\|t}}$ and $\frac{dM_{F\|t+dt}}{M_{F\|t}}$.[45]

By virtue of the projection (9), there is a unique pricing kernel that is linear in returns on assets $\overline{Y}_e$ and prices these assets. Apply this insight to $\left.\frac{dM_{H\|t+dt}}{M_{H\|t}}\right|_{\overline{\mathcal{R}}_e}$ and $\left.\frac{dM_{F\|t+dt}}{M_{F\|t}}\right|_{\overline{\mathcal{R}}_e}$ (both are linear in asset returns in $\overline{\mathcal{R}}_e$ by constructions (64)-(65), and price all such assets identically (70)), proves (66) ■

[45]This is because of the orthogonality in (62): asset $\overline{Y}_e\in\overline{\mathcal{R}}_e$ is not priced by (earning zero risk premium) by components $\left.\frac{dM_{H\|t+dt}}{M_{H\|t}}\right|_{\mathcal{R}_e}$ and $\left.\frac{dM_{F\|t+dt}}{M_{F\|t}}\right|_{\mathcal{R}_e}$ of SDF projectors which are only sensitive to the exchange rate risks.

Step S3: SDFs Projected on the Exchange Rate Risk Space

We next turn to the SDF projector components that are sensitive to the exchange rate risks. Intuitively, the exchange rate risks are completely disentangled, asset markets are complete regarding the exchange rate risks. Therefore, the complete-market standard identity between the exchange rate and the ratio of SDFs holds within this exchange rate risk space. Formally, we have,

Proposition 5 (Exchange Rate Risk Pricing) *When the exchange rate risks are completely disentangled, SDFs projected on the exchange rate risk space satisfy,*

$$\left.\frac{dM_{H\|t+dt}}{M_{H\|t}}\right|_{\mathcal{R}_e} = \left.\frac{d\left(e_{t+dt}M_{F\|t+dt}\right)}{e_t M_{F\|t}}\right|_{\mathcal{R}_e}. \tag{71}$$

Proof: Consider an asset $Y_i \in \mathcal{R}_e$ that loads singly on an exchange rate risk i of either diffusion or jump nature. Such Y_i exists because the set $\mathcal{R}_e$ is completely disentangled. We first make use of a general result concerning the no-arbitrage pricing across currencies, namely Proposition 1. Because this Proposition applies for any pricing kernel that prices all traded asset correctly in the home and foreign currencies, it holds for SDF projectors. Accordingly, with projectors $M_{H\|}$, $M_{F\|}$ being the pricing kernel in equation (43), applying that equation on every such asset Y_i (which loads only on one exchange rate risk i) yields,

$$\eta_{H\|i} = \eta_{F\|i} - \sigma_{ei}, \qquad \Delta_{H\|i} = \Delta_{F\|i} + \Delta_{ei}, \quad \forall i \in \mathcal{R}_e. \tag{72}$$

That is, system (43) is completely decoupled into a set of simple identities, each concerning only a single exchange rate risk $i \in \mathcal{R}_e$, when the exchange rate risks $\mathcal{R}_e$ are completely disentangled. Observe that identities (72) constitute the matching of respective diffusion and jump terms on left- and right-hand sides of (71) for every exchange rate risk $i \in \mathcal{R}_e$. Because either of these sides prices assets in the home currency consistently, the matching of drift terms (i.e., terms associated with dt) is then warranted (see Proposition 2). This proves identity (71) for every exchange rate risk i, and hence, proves Proposition 5 ■

Connecting the Dots

Aggregating results of Steps S1-S3 establishes the sufficient condition (61). First, by adding one to both sides of identity (71) of Proposition 5, we have,

$$\left(1+\frac{de_{t+dt}}{e_t}\right)\left(1+\left.\frac{dM_{F\|t+dt}}{M_{F\|t}}\right|_{\mathcal{R}_e}\right)=\left(1+\left.\frac{dM_{H\|t+dt}}{M_{H\|t}}\right|_{\mathcal{R}_e}\right).$$

Second, because non exchange rate risks $\overline{\mathcal{R}}_e$ are orthogonal to the exchange rate risks in $\frac{de}{e}$, we have the cross terms $\frac{de_{t+dt}}{e_t}\left.\frac{dM_{F\|t+dt}}{M_{F\|t}}\right|_{\overline{\mathcal{R}}_e}=0=\frac{de_{t+dt}}{e_t}\left.\frac{dM_{H\|t+dt}}{M_{H\|t}}\right|_{\overline{\mathcal{R}}_e}$. As a result,

$$\left(1+\frac{de_{t+dt}}{e_t}\right)\left.\frac{dM_{F\|t+dt}}{M_{F\|t}}\right|_{\overline{\mathcal{R}}_e}=\left(1+\frac{de_{t+dt}}{e_t}\right)\left.\frac{dM_{H\|t+dt}}{M_{H\|t}}\right|_{\overline{\mathcal{R}}_e}$$

Adding the last two equations yields,

$$\left(1+\frac{de_{t+dt}}{e_t}\right)\left(1+\left.\frac{dM_{F\|t+dt}}{M_{F\|t}}\right|_{\mathcal{R}_e}+\left.\frac{dM_{F\|t+dt}}{M_{F\|t}}\right|_{\overline{\mathcal{R}}_e}\right)=\left(1+\left.\frac{dM_{H\|t+dt}}{M_{H\|t}}\right|_{\mathcal{R}_e}+\left.\frac{dM_{H\|t+dt}}{M_{H\|t}}\right|_{\overline{\mathcal{R}}_e}\right),$$

which is

$$\frac{e_{t+dt}}{e_t}\frac{M_{F\|t+dt}}{M_{F\|t}}=\frac{M_{H\|t+dt}}{M_{H\|t}}.$$

This last identity proves the sufficient condition (61) (see also Remark 1, Appendix A.2).

B.2 The Necessary Condition

This section demonstrates that: assuming A1 and A2, if e_t is the ratio of SDF projectors, then every risk impacting the exchange rate e_t can be singly traded in financial markets,

$$e_t=\frac{M_{H\|t}}{M_{F\|t}},\quad \forall t\in[0,\infty)\quad\Longrightarrow\quad \text{Exchange rate risks are completely disentangled.}\tag{73}$$

Again, because the projectors (9) are constructed on the net growth quantities, the above equality rigorously presents, $\frac{de_{t+dt}}{e_t}=d\frac{M_{H\|t+dt}}{M_{F\|t+dt}}/\frac{M_{H\|t}}{M_{F\|t}}$ (Remark 1, Appendix A.2).

Plan of Attack: Starting from the assumed equality of the exchange rate and the ratio of SDF projectors, the derivation of this necessary condition for the Hypothesis H proceeds as follows.

N1. First, the key identity $e=\frac{M_{H\|}}{M_{F\|}}$ is equivalently transformed into a system of linear equations establishing $\frac{M_{H\|t}}{e_t}$ as a linear combination of asset returns $\{eY_n\}$ in the foreign currency.

N2. Second, when exchange rate risks are entangled, with probability one this system of linear equations has no solution, or equivalently the identity $e = \frac{M_{H\|}}{M_{F\|}}$ does not hold.

Therefore, the identity $e = \frac{M_{H\|}}{M_{F\|}}$ implies completely disentangled exchange rate risks, which proves the necessary condition (73). We now carry out this attack plan in detail.

Step N1: Constructing a Linear System

To quantify the viability of the identity $e = \frac{M_{H\|}}{M_{F\|}}$ in the necessary condition (73) of Theorem 1, we construct a system of linear equations which is equivalent to this identity.

To this end we linearly project the ratio of $\frac{M_{H\|}}{e}$ on the space of asset returns denominated in the foreign currency. That is, we look for weights β's in the following linear spanning involving the growths of the concerned quantities (see also Remark 1 for notation),[46]

$$\frac{\frac{M_{H\|t+dt}}{e_{t+dt}}}{\frac{M_{H\|t}}{e_t}} = \sum_{n=0}^{N} \beta_n \frac{Y_{Fnt+dt}}{Y_{Fnt}}. \tag{74}$$

Before addressing the viability of this linear projection, we make a simple observation on the equivalence between the identity $e = \frac{M_{H\|}}{M_{F\|}}$ and (74).

Proposition 6 *The linear spanning (74) exists if and only if the Hypothesis H on the asset market view of the exchange rate determination holds.*

Proof: First, we observe that the expression on the left-hand side of (74) is a consistent pricing kernel in the foreign currency. This is because projector $M_{H\|}$ is a consistent pricing kernel in the home currency. Indeed, this consistency can be seen in the explicit pricing of asset $Y_{Fn} \equiv eY_n$ in the foreign currency,

$$E_t \left[\frac{\frac{M_{H\|t+dt}}{e_{t+dt}}}{\frac{M_{H\|t}}{e_t}} \frac{Y_{Fnt+dt}}{Y_{Fnt}} \right] = E_t \left[\frac{\frac{M_{H\|t+dt}}{e_{t+dt}}}{\frac{M_{H\|t}}{e_t}} \frac{e_{t+dt}Y_{nt+dt}}{e_t Y_{nt}} \right] = E_t \left[\frac{M_{H\|t+dt}}{M_{H\|t}} \frac{Y_{nt+dt}}{Y_{nt}} \right] = 1, \quad \forall n.$$

Next, as a result of projection (9), there is a unique such pricing kernel in the foreign currency (that is linear in asset returns and prices these returns). Therefore, this pricing kernel must be identical to the foreign SDF projector, or $\frac{\frac{M_{H\|t+dt}}{e_{t+dt}}}{\frac{M_{H\|t}}{e_t}} = \frac{M_{F\|t+dt}}{M_{F\|t}}$, which proves the proposition ■

[46] Notice that the linear spanning is not necessarily a portfolio representation, therefore it is possible that sum of weights differs from unity, $\sum_{n=0}^{N} \beta_n \neq 1$. For simplicity of notation, the term corresponds to $n = 0$ in (74) is associated with the foreign risk-free bond B_F.

The above result simply allows us to equivalently substitute the necessary condition (73) of Theorem 1 by the viability of expressing the ratio $\frac{M_{H\|}}{e}$ as a linear combination of asset returns in the foreign currency, i.e., the spanning (74). Observe that (74) is linear system of unknown weights $\{\beta_n\}$ and equations (matching of every independent risk entering the left- and right-hand sides of (74)).[47] Crucially, Proposition 6 hence suggests standard counting arguments concerning the number of the unknowns and constraints underlying (74) to assess its viability as a way to establish the necessary condition (73).

Step N2: Viability of the Linear System

The linear equation system (74) has $N+1$ unknown $\{\beta_n\}$, $n \in \{0, \ldots, N\}$ (which is number of traded assets in the international markets). To discern the number of equations, (which is the dimension of risks impacting the system), let $\mathcal{R}_\beta$ denote the set of risks that enter the linear system (74). The following result establishes risk entanglement as a sufficient condition for the non-viability of (74).

Proposition 7 *When risks $\mathcal{R}_\beta$ impacting the system (74) of linear equations are entangled, the system is almost surely inconsistent and hence has no solution with probability one.*

Proof: At an intuitive level, the idea behind the above result is a straightforward counting argument. By the nature of the risk entanglement (Definition 1), when the risk set $\mathcal{R}_\beta$ is entangled, the set of all traded assets is insufficient to singly replicate every risk in $\mathcal{R}_\beta$, therefore,

$$\text{Risk set } \mathcal{R}_\beta \text{ is entangled} \quad \Longrightarrow \quad N+1 < dim(\mathcal{R}_\beta). \tag{75}$$

Because (74) is a linear system of $N+1$ unknowns $\{\beta_n\}$ and $dim(\mathcal{R}_\beta)$ linear equations (each equation is a matching condition for one risk in set $dim(\mathcal{R}_\beta)$), the strict inequality in (75) implies that the linear system (74) is almost surely inconsistent (having more equations than unknowns).[48] Continued with our intuition, the result in (75) simply implies that when the risks in $\mathcal{R}_\beta$ are entangled, with probability one the linear system (74) has no solution ■

[47] Recall that the system (74) is the quest to express $\frac{M_{H\|}}{e}$ linearly in returns on Y_{Fn}, i.e., weights β_n are sought-after quantities (or unknowns) in that equation system, given $M_{H\|}$, e and $\{Y_{Fn}\}$ processes.

[48] A technical and complete derivation of this intuitive counting argument makes use of the Sard's theorem to rigorously address the measure-zero event of redundancy of equations.

Connecting the Dots

Aggregating results of Steps N1-N2 establishes the necessary condition (73). Specifically, note that risks $\mathcal{R}_\beta$ contributing to (74) are the exchange rate risks and non-exchange rate risks.[49] Therefore, exchange rate risks $\mathcal{R}_e$ is a subset of risks entering (74); $\mathcal{R}_e \subseteq \mathcal{R}_\beta$. Consequently, if exchange rate risks $\mathcal{R}_e$ are entangled, then risks in $\mathcal{R}_\beta$ must also be entangled.[50] Combining this result with that of Proposition 7 implies that if the exchange rate risks $\mathcal{R}_e$ are entangled, then with probability one the linear system (74) has no solution. Equivalently, we have,

$$\text{Solution to system (74) exists} \implies \text{the exchange rate risks are completely disentangled.}$$

Together with the equivalence between the viability of the linear system (74) and Hypothesis H reported in Proposition 6, the above conclusion implies that, if Hypothesis H holds (i.e., $e = \frac{M_{H\parallel}}{M_{F\parallel}}$, see also Remark 1 in Appendix A.2), then the exchange rate risks are completely disentangled. This proves the necessary condition (73) of Theorem 1.

B.3 Theorem 2 and A Look-Back on Risk Entanglement

We observe that the notion of exchange rate risk entanglement has another equivalent interpretation that is very intuitive. To discern such an interpretation we recall that in absence of risk entanglement (e.g., pure-diffusion risk settings), spaces of asset returns denominated in the home and foreign currencies are identical. That is, any traded return in the home currency can be literally and linearly spanned by asset returns in the foreign currency.[51] This picture changes completely in the presence of risk entanglement in the exchange rate as the following result demonstrates.

Proposition 8 *The exchange rate risks are completely disentangled in asset markets if and only if the asset return spaces in the home and foreign currencies are identical,* $\mathcal{R} = \mathcal{R}_F$.

[49] The exchange rate risks $\mathcal{R}_e$ enter the equation system via the explicit presence of exchange rate e in (74). Non exchange rate risks may also enter the equation system via asset returns in either currencies. This is possible because an asset may load on several types of risks a priori.

[50] The proof is by contradiction, as follows. Contrarily, assume that $\mathcal{R}_\beta$ is completely disentangled (and $\mathcal{R}_e \subset \mathcal{R}_\beta$ and $\mathcal{R}_e$ is entangled). Then by the virtue of complete disentanglement of risks $\mathcal{R}_\beta$, there is at least an asset loading singly on every risk in $\mathcal{R}_\beta$, and thus every risks in $\mathcal{R}_e$ as well because $\mathcal{R}_e \subset \mathcal{R}_\beta$. But the last result implies that $\mathcal{R}_e$ is completely disentangled, which contradicts the assumption.

[51] This spanning is mechanical, because it does not involve the exchange rate factor. To see this, note that in the absence of risk entanglement, multiplying with exchange rate does not alter the risk space, as the familiar case of pure diffusion risk setting illustrates.

Proof: First, completely disentangled exchange rate risks implying the currency-neutrality (i.e., currency-independence) of the asset return space (the sufficient condition of this Proposition) is given by the orthogonal decomposition result of Proposition 3.

Second, when asset return spaces in the home and foreign currencies are identical, (recall our simplified notation $\mathcal{R}_F \equiv e\mathcal{R}$), we simply have $\mathcal{R} = e\mathcal{R}$. By iteration, therefore $\mathcal{R} = e^k\mathcal{R}$ for any finite integer $k > 0$, in which $e^k\mathcal{R}$ denotes the set of asset returns of the form $\frac{d(e^k_{t+dt}X_{t+dt})}{e^k_t X_t}$, with X being an asset in the original set $\mathcal{R}$. As a result, $\mathcal{R}$ is identical to the composite asset return space $\left\{\{\mathcal{R}\} \cap \{e\mathcal{R}\} \ldots \cap \{e^k\mathcal{R}\}\right\}$ for any finite integer $k > 0$. This then implies that the exchange rate risks are completely disentangled in asset markets ■

The combination of Theorem 1 and Proposition 8 immediately offers an alternative necessary and sufficient condition for Hypothesis H reported in Theorem 2.