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The Functional Stochastic Discount Factor

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By assuming that the stochastic discount factor (SDF) M is a proper but unspecified function of state variables X, we show that this function $M(X)$ must solve a simple second-order linear differential equation specified by state variables' risk-neutral dynamics. Therefore, this assumption determines the most general possible SDFs and associated preferences, that are consistent with the given risk-neutral state dynamics and interest rate. A consistent SDF then implies the corresponding state dynamics in the data-generating measure. Our approach offers novel flexibilities to extend several popular asset pricing frameworks: affine and quadratic interest rate models, as well as models built on linearity-generating processes. We illustrate the approach with an international asset pricing model in which (i) interest rate has an affine dynamic term structure and (ii) the forward premium puzzle is consistent with consumption-risk rationales; the two asset pricing features previously deemed conceptually incompatible.

Keywords: Stochastic discount factor; dynamic term structure models; linear generating processes; forward premium puzzle.

JEL Classification: G00, G12, D84

1. Introduction

Stochastic discount factor (SDF) is one of the most conceptually important objects in asset pricing theory and modeling. From a pure pricing perspective, the existence of SDF is equivalent to absence of arbitrages. From an equilibrium perspective, SDF is the marginal utility of representative agents in the associated (complete-market) economy. From a modeling perspective, many asset pricing models are just set out with an SDF specification. Despite these conceptual prominences, a fundamental issue with SDF is that it is not directly observed in markets.

In the current paper, we propose a novel and tractable asset pricing construction in which the SDF is assumed to be some (unspecified) function of the underlying state variables. We then demonstrate that, in continuous time and state settings, every functional SDF satisfies a linear second-order differential equation parametrized by the interest rate and risk-neutral dynamics (i.e., drift and volatility) of state variables. Therefore, the functional SDF assumption suffices to determine the most general possible SDFs that are consistent with the given risk-neutral state dynamics and interest rate. By construction, functional SDFs fit naturally into the utilitarian framework by identifying state variables with consumption and other quantities, that influence economic agents' utilities. The utilitarian framework in turn provides economic interpretations for functional SDFs.

Our paper develops a functional SDF approach along the following line. First, we introduce a basic construction of functional SDFs from risk-neutral state dynamics, and discuss its economic and modeling merits. Second, we extend the basic construction to deliver functional SDFs from state dynamics in any equivalent measure, and recover popular (affine and quadratic) and obtain new (fractional) dynamic term structure models (DTSMs) of interest rate. Third, we observe that asset pricing models, built on linear-generating dynamics, belong to the functional SDF approach, and generalize the former based on this observation. Finally, we illustrate functional SDFs in an international setting, in which the forward premium puzzle (FPP) is consistent with an affine consumption dynamics inherent in these functional SDFs. We elaborate on these results next.

First, when SDF is a function of state variables, the functional form is constrained by the dynamics of these state variables. In risk-neutral measure, the pricing of a risky payoff (i.e., the role of SDF) is reduced to discounting the payoff by risk-free rate (rfr). Therefore, risk-neutral state dynamics and rfr suffice to establish a functional constraint on the SDF, from which all consistent SDFs can be determined.[1] That is, a change of measure from risk-neutral to physical measure systematically maps rfr and risk-neutral state dynamics into possible SDFs. Equilibrium asset pricing models can be broadly classified into two groups. In the first, SDF is identified structurally

[1]Specifically, the expected growth (drift) of SDF equals the additive inverse of rfr. For functional SDF, its drift is functionally related to the dynamics (drift and volatility) of state variables via Itô's lemma. Identifying this drift with the additive inverse of rfr produces a differential equation for the functional SDF.

with a representative agent's marginal utility of consumption and thus, is motivated fundamentally by the rational time and risk preferences of market participants. Known examples include consumption-based capital asset pricing models. In the second, SDF is specified in a reduced form, together with risk-neutral state dynamics, to deliver closed-form asset prices. Known examples include affine DTSMs of interest rate. Our basic construction is motivated to bridge the two groups. Functional SDFs, when properly constructed, can have both the economic interpretation of the first group and the pricing tractability of the second.

Second, we observe that asset pricing can be performed in any equivalent measure, given state and pricing kernel dynamics in that measure. Therefore, we can start with reduced-form state dynamics in an equivalent measure (not necessarily risk-neutral or physical) that deliver closed-form asset prices. A change of measure similar to the basic construction above, this time from the equivalent measure to physical measure, generates a second-order differential equation that determines all possible functional SDFs. The introduction of an equivalent measure enriches the basic functional SDF construction with new flexibilities. In particular, even when state dynamics are affine in the equivalent measure, they become non-affine in both physical and risk-neutral measures. As a result, we recover not only standard affine and quadratic DTSMs, but also new tractable fixed-income pricing models in which state dynamics and rfr are fractional functions of state variables in physical and risk-neutral measures.

Third, the recently proposed tractable asset pricing class, based on linearity-generating (LG) processes, also fits well into our construction. This is because SDF is a (linear) function of state variables in LG settings. We observe that the LG class of asset pricing models possesses a measure-invariant property. Namely, if the model is LG in a measure, it remains LG in any other equivalent measure. This property, hence, deprives the LG class of generalizations that employ a change of measure discussed above. However, built upon our differential approach, we are able to construct a more general version of LG pricing models that does not have to set out with strictly LG processes.

Finally, the advantage of the functional SDF construction is illustrated and employed to reconcile the FPP in international finance. FPP is a puzzling empirical regularity in which currencies tend to appreciate when the associated interest rates also increase.[2] We consider an international asset pricing

[2] This is puzzling because it appears that appreciating currencies are more valuable, yet investors require higher premia (i.e., interest rates) to hold them.

model, in which the functional SDF is a power and exponential (hybrid) function of consumption. When coupled with mean-reverting consumption dynamics and affine rfr, the resulting price of consumption risk correlates negatively with rfr. Consequently, the appreciation in the exchange rate moves in the same direction as the interest rates' differential, a pattern that is consistent with the FPP. Intuitively, when the home economic condition improves and consumption surges, home risk-free bonds lose their appeal, become cheaper, and the home interest rate increases. At the same time, investors also perceive lower risks in the home market, loosen their risk-based discounting, and require lower prices of the consumption risk. They end up valuing the home currency more favorably. Altogether, the model features an affine dynamic term structure of interest rates and a consumption-base rationale for the FPP: the home currency tends to appreciate when the home interest rate increases (relative to the foreign counterparts).

Our paper contributes to the literature of structural asset pricing models that also deliver tractable asset prices. Constantinides (1992) and Rogers (1997) directly formulate SDF in physical measure without invoking associated utilities. In their works, SDFs are exogenously specified. In comparison, our construction assumes that SDFs are unspecified functions of state variables. We then determine the functional SDF systematically and consistently from a differential equation, parametrized by state dynamics. In reduced-form tractable asset pricing literature, Duffie and Kan (1996) construct a general class of affine risk-neutral dynamics that encompasses several classic bond pricing models. The current paper generalizes this modeling paradigm by flexibly introducing affine dynamics in any equivalent measure. Dai and Singleton (2000) construct a general scheme to classify and analyze DTSMs of interest rate, based on enumerating relevant volatility factors and their invariant transformations (i.e., symmetries). The current paper also attempts to classify term structure models, but based on a very different dimension. In our scheme, different models are related if their dynamics can be rotated from one to the other by a change of measure. In this classification, affine, quadratic, and quotient DTSMs are variants of affine dynamics in some equivalent measure. More recently, Gabaix (2009) introduces the class of LG processes, in conjunction with a linear SDF specification that delivers tractable bond and stock prices. The current paper generalizes his construction by incorporating general (non-LG) dynamics of underlying state variables.

Our paper is related to Ross (2015), who derives sufficient conditions to recover unique physical state dynamics and preference from risk-neutral dynamics in a discrete time and state setting. Carr and Yu (2012) and

Walden (2017) generalize the unique recovery result to bounded and unbounded continuous time and state settings. However, Hansen and Scheinkman (2009) and Borovicka *et al.* (2016) show that a sufficient condition needed for the unique recovery of preference also has counterfactual implications on asset prices that are not supported by price data (see also Bakshi *et al.* (2017)).[3] Tran and Xia (2018) show that the discretization process, in the implementation of the unique recovery, is fragile and susceptible to mis-specifications. The current paper's functional SDF approach imposes weaker conditions and delivers many possible SDFs consistent with the given risk-neutral dynamics. *Ex-post*, additional conditions are needed to pick out a desired SDF from these solutions. Hence, while this feature does not deliver a unique recovery, it gives flexibility to reconstruct empirically relevant asset pricing models.

The forward premium puzzle (FPP; Fama (1984)) is investigated within tractable bond pricing frameworks by Backus *et al.* (2001), who show that it is difficult to reconcile affine state dynamics and FPP. Our functional SDF approach identifies a generalized SDF of the (hybrid) exponential and power function of consumption, that enables forward premium anomaly which, by construction, is also consistent with affine state dynamics.

The paper is structured as follows: Section 2 introduces the set-up and motivations of functional SDFs, and relates them to (Ross, 2015)'s recovery theorem. Section 3 presents various generalizations of the basic functional SDF construction. Section 4 relates functional SDFs to LG processes. Section 5 discusses an international asset pricing model featuring functional SDFs, which is consistent with both the FPP and an affine term structure of interest rate. Section 6 concludes. Appendices present technical derivations omitted in the main text.

2. Construction of SDF

In this section, we present a general construction of SDFs when they are presumably functions of underlying state variables. We begin with the formal construction, and discuss various motivations for the construction.

[3] Specifically, Borovicka *et al.* (2016) note that Ross (2015)'s unique recovery assumes purely transitory SDFs (i.e., having no permanent component). But because SDF transitory components can be identified with long bond returns (Alvarez and Jermann, 2005), long bond returns necessarily perfectly correlate with SDF, and hence, necessarily offer the largest Sharpe ratio among all traded assets in a unique recovery premise. The last implication however is counterfactual.

2.1. *Set-up*

To set the notation, we first consider a basic stochastic setting driven by a state variable $X(t)$ of diffusion dynamics in either risk-neutral (Q) or physical (P) measure,

$$\begin{aligned} dX(t) &= \mu^{X,Q}(X,t)\,dt + \sigma^X(X,t)\,dZ^Q(t) \\ &= \mu^{X,P}(X,t)\,dt + \sigma^X(X,t)\,dZ^P(t), \end{aligned} \tag{1}$$

where $Z(t)$ denotes a standard Brownian motion in the respective measure, and μ^X and σ^X denote the adaptive drift and volatility processes. For simplicity, we first assume that $X(t)$ is a scalar process, and hereafter omit the state and time dependence notation (X,t) whenever possible.[4] Let $M^P(X,t)$ denote the SDF process in the physical measure,

$$\frac{dM^P}{M^P} = -r(X,t)\,dt - \eta^{QP}(X,t)\,dZ^P(t), \tag{2}$$

where r and η^{QP} denote the rfr and price of risk, respectively. The change between physical and risk-neutral measures, $dZ^Q(t) = dZ^P(t) + \eta^{QP}dt$, is characterized by the Radon–Nikodym derivative η^{QP},

$$\xi^{QP}(t) = \exp\left(\int^t -\frac{1}{2}(\eta^{QP})^2(X,s)\,ds - \eta^{QP}(X,s)\,dZ^P(s)\right). \tag{3}$$

2.2. *Construction: Basic version*

The construction is motivated by the risk-neutral pricing methodology and starts with the state Q-dynamics $\mu^{X,Q}$, σ^X and rfr process r. This specification is customary in the DTSM of interest rate, and may also be inferred from observed derivative prices (Breeden and Litzenberger, 1978). We make the following key assumption in the SDF construction.

Assumption 1. In the physical measure P, SDF M^P is a proper function of the state variable X and time t in $M^P(X,t) = e^{-\rho t}M^P(X)$, where the constant parameter ρ denotes the time discount factor.[5]

Note that the above assumption neither specifies nor imposes any specific functional form for $M^P(X)$. Instead, the most general and consistent

[4] Multi-dimensional state variables are the topic of Appendix B. The inclusion of jump processes is also possible.

[5] The assumption of constant time discount factor ρ is convenient but nonessential for our analysis. Extension to the case where ρ is some function of time is possible, but conceptually contributes little to the construction.

$M^P(X)$ can be consistently determined. To see this, we first express the SDF process (2) in risk-neutral measure

$$dM^P(X,t) = -M^P(X,t)[\{r(X,t) - (\eta^{QP}(X,t))^2\}dt + \eta^{QP}(X,t)dZ^Q(t)],$$

and then identify it with the stochastic differential representation of dM^P obtained from applying Itô's lemma on the function $M^P(X,t)$. The matching of the drift and diffusion terms yields,

$$\frac{1}{2}(\sigma^X(X,t))^2 M^P_{XX}(X) + \mu^{X,Q}(X,t)M^P_X(X) + [r(X,t) - (\eta^{QP}(X,t))^2 - \rho]M^P(X) = 0, \tag{4}$$

$$\sigma^X(X,t)M^P_X(X) + \eta^{QP}(X,t)M^P(X) = 0, \tag{5}$$

and their combination produces a single differential equation for the SDF,

$$\frac{(\sigma^X(X,t))^2}{2}M^P_{XX}(X) + \mu^{X,Q}(X,t)M^P_X(X) + \left(r(X,t) - \frac{(\sigma^X(X,t))^2(M^P_X(X))^2}{(M^P(X))^2} - \rho\right)M^P(X) = 0. \tag{6}$$

In principle, rfr $r(X,t)$, and Q-dynamics $\{\mu^{X,Q}(X,t), \sigma^X(X,t)\}$ suffice to determine possible SDFs M^P that satisfy Assumption 1. In practice, (6) is a highly non-linear differential equation, whose solution can be elusive.

A careful observation helps to address this non-linearity. We note that the Radon–Nikodym derivative $\xi^{PQ} = \frac{\exp(-\int^t rds)}{M^P(t)}$ is a Q-martingale, which is linear in the inverse of SDF $\frac{1}{M^P(t)}$. This hints at the linearity of the construction in $\frac{1}{M^P}$ and suggests a change of variable,

$$\phi^P(X) \equiv \frac{1}{M^P(X)} = \frac{e^{-\rho t}}{M^P(X,t)}. \tag{7}$$

In fact, (6) becomes a homogeneous second-order linear differential equation (HSOLDE) in $\phi^P(X)$,[6]

$$\frac{1}{2}(\sigma^X(X,t))^2\phi^P_{XX}(X) + \mu^{X,Q}(X,t)\phi^P_X(X) + [\rho - r(X,t)]\phi^P(X) = 0. \tag{8}$$

[6] Note that we need to impose the appropriate condition $\phi^P(X) \geq 0\ \forall X$, and a conventional initial condition $\phi^P(X(0)) = 1$ for M^P to be a proper SDF. Since these boundary conditions are on a case-by-case basis, we omit further details in the current section's general discussion. Cheridito *et al.* (2007) explicitly treat the regularity conditions for the class of extended affine DTSM. The Feller's admissibility condition for the square-root process (a.k.a., Cox–Ingersoll–Ross or CIR) is discussed in Sec. 5.2.

From (2) and (8) follow consistently the market price of risk (mpr) η^{QP} and the P-dynamics $\{\mu^{X,P}(X,t)\}$

$$\begin{aligned}\eta^{QP}(X,t) &= \frac{-M_X^P(X,t)}{M^P(X,t)}\sigma^X(X,t) = \frac{\phi_X^P(X,t)}{\phi^P(X,t)}\sigma^X(X,t),\\ \mu^{X,P}(X,t) &= \mu^{X,Q}(X,t) + \eta^{QP}(X,t)\sigma^X(X,t).\end{aligned} \tag{9}$$

We recapitulate this basic construction of the SDF in the following proposition.

Proposition 1. *Given state dynamics* $\{\mu^{X,Q}(X,t), \sigma^X(X,t)\}$ *in risk-neutral measure* Q *and rfr* $r(X,t)$, *if the SDF in physical measure* P *is a proper function of state variable* $M^P = e^{-\rho t}M^P(X)$ *then* $\phi^P(X) \equiv \frac{1}{M^P(X)}$ *solves the second-order linear differential* (8). *Under this condition, physical dynamics* $\mu^{X,P}(X,t)$ *and mpr* η^{QP} *can be consistently inferred as in* (9).

Proof. In place of the above intuitive argument leading to Proposition 1, we sketch here a direct proof. The Radon–Nikodym derivative $\xi^{PQ} = \frac{\exp(-\int^t r(X,s)\,ds)}{M^P(X,t)}$ can be written as $\exp(-\int^t [r(X,s) - \rho]\,ds)\phi^P(X)$. It is a Q-martingale, and so has no drift term under the risk-neutral measure. Assumption 1 then allows us to obtain an explicit expression of ξ^{PQ}'s drift under Q-measure in terms of $\{\mu^{X,Q}(X,t), \sigma^X(X,t)\}$. Identifying this drift with zero immediately gives rise to Eq. (8), and the above proposition. □

The most remarkable feature of the basic SDF construction lies in its linearity. Starting out with the observable risk-neutral dynamics $\{\mu^{X,Q}(X,t), \sigma^X(X,t)\}$ and rfr $r(X,t)$, we can determine the most general SDF M^P, mpr η^{QP}, and physical state dynamics $\{\mu^{X,P}(X,t), \sigma^X(X,t)\}$. In our current continuous-state approach, any specific and qualified[7] solution of (8) may constitute a possible SDF, consistent with the same observable risk-neutral dynamics $\{r(X,t), \mu^{X,Q}(X,t), \sigma^X(X,t)\}$. This substantially simplifies the application, and hence empowers the functional SDF approach. Instead of solving this differential equation in earnest generality, we can just construct a specific solution with appropriate properties motivated by economic considerations. The obtained SDF then is consistent with both the prescribed Q-dynamics and economic considerations. This key feature also holds for the multi-factor functional SDF construction of Appendix B.

[7] The strict positivity of SDFs is required to enforce no arbitrage. Other properties may also be motivated and imposed out of economic considerations.

2.3. *In relation to the recovery theorem*

In a related work, Ross (2015) formulates a recovery theorem, i.e., a set of sufficient conditions to recover the SDF and state dynamics in the physical measure from rfr and state dynamics in the risk-neutral measure.[8] Ross (2015)'s recovery theorem employs an algebraic (matrix) approach in a discrete state space setting, and features a unique recovery of preference and physical state dynamics. In comparison, the construction underlying Proposition 1 employs an analytical (differential equation) approach in a continuous state space setting, and obtains the most general, but not unique, SDFs that are consistent with the given rfr and state dynamics in the risk-neutral measure. Naturally, our sufficient condition (Assumption 1) is weaker than Ross (2015)'s sufficient condition for a unique recovery. This section reconciles the two approaches by showing that both arise from the same transition probability dynamics in the risk-neutral measure.

We start with a Q-martingale property $E_t^Q[\xi^{PQ}(T)] = \xi^{PQ}(t)$ of the Radon–Nikodym derivative $\xi^{PQ}(t) = \exp(-\int^t [r-\rho]\, ds)\phi^P(X)$, which implies

$$E_t^Q[e^{-\int_t^T (r-\rho)\, ds}\phi^P(Y)] = \phi^P(X).$$

To bring the above martingale condition to the formulation of the recovery theorem, we consider an infinitesimal period $T = t + dt$ and denote $p(X, t; Y, T)$ the transition probability density from (X, t) to (Y, T) in risk-neutral measure. The above expectation then can be written explicitly as

$$\int_Y e^{-\int_t^T (r-\rho)\, ds} p(X, t; Y, T)\phi^P(Y)\, dY = \phi^P(X). \tag{10}$$

This equation is the continuous version of the characteristic root equation in Ross (2015), and is key to the construction of the recovery theorem therein.[9] On the other hand, by employing the Kolmogorov backward equation on the risk-neutral transition probability density $p(X, t; Y, T)$,[10] and taking the

[8] The first draft of the current paper appeared in May 2011, and was developed simultaneously and independently of Ross (2015). Our draft did not deliver an unique SDF. We instead aimed to identify all possible SDFs that are consistent with the given state dynamics under a mild condition of "functional SDF" (Assumption 1).

[9] In the one-period discrete-state setting, $T = t + 1$, (10) becomes $\sum_j e^{-r_i} p_{ij}\phi_j = \delta\phi_i$, or the characteristic root equation $P\phi = \delta\phi$, where $P_{ij} = e^{-r_i} p_{ij}$; $\delta = e^{-\rho}$; $\phi = \{\phi_i\}_i$. To obtain the recovery theorem, Ross (2015) ensures the applicability of the Perron–Frobenius theorem on this equation by positing a static state space structure.

[10] The Kolmogorov equation is $-\frac{\partial}{\partial t} p(X, t; Y, T) = \mu^{X,Q}(X, t)\frac{\partial}{\partial X} p(X, t; Y, T) + \frac{1}{2}(\sigma^X(X, t))^2 \frac{\partial^2}{\partial X^2} p(X, t; Y, T)$,

partial derivatives with respect to t and X of both sides of (10), yields the key differential equation (8) that underlies Proposition 1. Therefore, in technical terms, both the recovery theorem and the current paper's SDF construction trace their roots back to a fundamental change of measure, from given risk-neutral dynamics to to-be-determined physical dynamics. While not producing unique physical dynamics, our construction instead systematically determines a set of all possible functional SDFs that are consistent with the given risk-neutral dynamics. This flexibility helps us to identify, among these possible SDFs, the one with desired economic properties, and reverse engineer an equilibrium model consistent with the desired SDF. Section 5 presents an international asset pricing model constructed in this manner to address the FPP.

2.4. *Motivations and discussion*

We discuss several notable features of the SDF construction of Proposition 1. First, the assumption that the SDF is a function of state variables (Assumption 1) features predominantly in consumption-based equilibrium asset pricing models. Therein, the SDF is the representative agent's marginal utility, and is some function $M^P = \frac{\partial U(C,H)}{\partial C} = U_C(C, H)$ of aggregate consumption C, and possibly other state variables such as consumption surplus H in habit formation settings. This is one of the motivations of our SDF construction to place certain no-arbitrage pricing models, e.g., some DTSM of interest rates, on a utilitarian framework. In this regard, although the construction generates restrictions on the market price risk η^{QP} (5), we have the freedom in specifying interest rate function $r(X, t)$. In turn, the resulting physical state dynamics $\mu^{X,P}(X, t)$ (9) are rich. In the next section, we consider a class of tractable bond pricing models, wherein rfr $r(X, t)$, state dynamics $\mu^{X,P}(X, t)$, and even $\mu^{X,Q}(X, t)$ are not affine in X. The class, thus, is beyond the affine DTSM framework.

Second, Assumption 1 on function $M^P(X, t)$ appears similar to imposing a Markovian structure on SDFs. Under standard conditions, the diffusion dynamics (1) imply that $X(t)$ is Markovian, and so is the regular function $M^P(X, t)$. In general cases, SDFs depend on an entire historical path of state process $\{X(s)\}_0^t$.[11] Therefore, given exogenous and arbitrary functions $r(X, t)$ and $\eta^{QP}(X, t)$, SDFs may not always be proper functions of solely the current state (X, t). It is only when $r(X, t)$ and $\eta^{QP}(X, t)$ are jointly

[11] To see this, recall that SDF has the following integral representation, $M^P = \exp - \int^t [\{\frac{1}{2}\eta^{QP}(X, s)^2 + r(X, s)\} ds + \eta^{QP}(X, s) dZ^P(s)]$.

constrained by the systems of equations (4) and (5), that SDF becomes a proper function of current state variable.[12] In other words, $\eta^{QP}(X,t)$ is implied from the fundamentals $\{\mu^{X,Q}(X,t), \sigma^X(X,t), r(t,X)\}$ via Assumption 1, and in the same process, a consistent SDF $M^P(X,t)$ is implied.[13]

Third, explicit functional SDFs $M^P(X,t)$ can facilitate testing and estimation via a generalized method of moments (GMMs). Especially, when underlying state variable X is observable, the associated Euler equation can be estimated in discrete time, following the standard procedure of Hansen and Singleton (1982):

$$E_t^P\left[\frac{M^P(X(t+1),t+1)}{M^P(X(t),t)}R(X(t+1),t+1)\right]=1,$$

where R is a gross return on any traded asset. Alternatively, the resulting physical dynamics $\{\mu^{X,P}(X,t), \sigma^X(X,t)\}$, explicitly obtained in this construction, are sufficient statistics to carry out an approximate but efficient maximum likelihood estimation, as proposed by Ait-Sahalia (2002).

Finally, we will analytically solve the key equation (8) for several important cases of risk-neural dynamics $\mu^{X,Q}$, $\sigma^{X,Q}$ and r in the next section. However, there exists a simple numerical solution method that works for arbitrary time-homogeneous dynamics $\mu^{X,Q}(X)$, $\sigma^{X,Q}(X)$, and $r(X)$. Therein, a simple change of variable $\psi^P(X)\equiv\frac{\phi_X^P(X)}{\phi^P(X)}$ transforms the second-order differential equation (8) into a first-order Riccati differential equation

$$\psi_X^P(X)+(\psi^P(X))^2+\frac{2\mu^{X,Q}(X)}{(\sigma^X(X))^2}\psi^P(X)+\frac{2[\rho-r(X)]}{(\sigma^X(X))^2}=0,$$

which can be numerically solved. By virtue of (5) and (7), $\psi^P(X)\equiv\frac{\phi_X^P(X)}{\phi^P(X)}=\frac{\eta^{QP}(X)}{\sigma^X(X)}$, and above Riccati equation directly determines mpr $\eta^{QP}(X)$.

[12] Indeed, given the unrelated functions $r(t,X)$ and $\eta^{PQ}(t,X)$, the systems (4) and (5) will generally have no solution $\phi^P(X)$ (or P-SDF $M^P(t,X)=\frac{e^{-\rho t}}{\phi^P(X)}$) because either $r(t,X)$ or $\eta^{PQ}(t,X)$ alone suffices to yield a solution $\phi^P(X)$ up to constants of integration.

[13] The functional SDFs do not rule out a path-dependent feature of M^P (which is useful in certain modeling premises because a Markovian SDF might have undesirable implications from an empirical perspective, Ross (2015) and Borovicka *et al.* (2016)). A simple illustration is when the state $X(t)$ itself depends on the entire path $\{Z^P(s)\}_0^t$ of Brownian motion $Z^P(t)$, and so does $M^P(X,t)$. As suggested in Chen and Joslin (2012), we may augment the state space to multi-dimensional settings (Appendix B) to absorb the path dependence of one variable into another new state variable. Alternatively, we can also embed our basic functional SDF construction in any other equivalent measure R (not necessarily P or Q). As a result, when returning to physical measure P, the SDF $M^P=M^P(\{X(s)\}_0^t,t)$ depends on the entire history path of the state variable (Construction 2 below).

Therefore, the mpr η^{QP} satisfies a simple Riccati differential equation under time-homogeneous risk-neutral dynamics. The next section applies the functional SDF construction to a DTSM of interest rate.

3. Affine Term Structure Modeling and Beyond

We now explore the integration of the functional SDF construction into affine and other DTSMs that possess remarkable pricing tractability. At a basic level, DTSM framework and functional SDF construction share the specification of risk-neutral dynamics $\{\mu^{X,Q}(X,t), \sigma^X(X,t)\}$ and rfr $r(X,t)$. This section substantially generalizes the basic SDF construction by specifying the state dynamics in an equivalent measure R, and recovers bond pricing tractability, while allowing for rich P- and Q-dynamics and non-affine rfr.

3.1. *Construction* 1: *Basic Q-dynamic term structure modeling*

We recall the key ingredients of affine DTSMs of interest rate (Vasicek, 1977; Cox *et al.*, 1985; Duffie and Kan, 1996) that feature tractable bond prices and yields. They are affine risk-neutral dynamics ($\mu^{X,Q}$ and $(\sigma^X(X))^2$ linear in X) and affine rfr (r linear in X). From these, follow prices of zero-coupon bonds in closed form and an affine term structure of interest rate.[14] Although leading affine DTSM models, such as completely affine (Dai and Singleton, 2000), essentially affine (Duffee, 2002), and extended affine (Cheridito *et al.*, 2007) also impose affine dynamics in physical measure P out of econometrics conveniences, affine Q-dynamics are key for bond pricing tractability.

Our first construction is built on Assumption 1 of functional SDFs, and affine risk-neutral dynamics to retain the bond pricing tractability.

Construction 1. *Assume the following*:

- *SDF is a proper, but unspecified, function of the state variable*: $M^P(X,t) = e^{-\rho t}M^P(X)$ (*Assumption* 1),
- *affine Q-dynamics*: $\mu^{X,Q}(X) = K_0^Q + K_1^Q X$; $(\sigma^X(X))^2 = H_0 + H_1 X$,
- *affine rfr*: $r(X) = a + bX$.

Note that Construction 1 is a special case of the basic construction underlying Eq. (8), in which both risk-neutral dynamics and rfr are given affine functions of state variable.

[14] The price of the zero-coupon bond of maturity T is $\mathrm{ZCB}(t,t+T) = E_t^Q[\frac{\exp -\int^{t+T} r(X(s))ds}{\exp -\int^{t} r(X(s))ds}] = \exp[A(T) + B(T)X(t)]$, where $A(T)$, $B(T)$ satisfy a system of Riccati equations.

3.1.1. *Implied SDFs*

As a special case of Eq. (8), affine Q-dynamics yield a differential equation on $\phi^P = \frac{1}{M^P}$,

$$\frac{1}{2}[H_0 + H_1 X]\phi^P_{XX}(X) + [K^Q_0 + K^Q_1 Xs]\phi^P_X(X) + [\rho - a - bX]\phi^P(X) = 0. \tag{11}$$

Appendix A derives the following analytical solution to this equation.

Proposition 2. *The most general functional SDF M^P consistent with Construction 1 is*

$$M^{P,\{\lambda_1,\lambda_2\}}(X,t) = \frac{e^{-\rho t}e^{-\frac{\alpha}{\beta}z}}{\lambda_1\Phi(\delta,\gamma;z) + \lambda_2 z^{1-\gamma}\Phi(\delta-\gamma+1, 2-\gamma; z)}, \quad z \equiv \beta(H_0 + H_1 X), \tag{12}$$

where λ_1, λ_2 are two constants of integration associated with differential equation (11), *$\alpha, \beta, \gamma, \delta$ are constant coefficients related to the model's parameters given in Appendix A, and $\Phi(\cdot,\cdot;z)$ is the confluent hypergeometric function of argument z.*

We note that there may be many functional SDFs consistent with the same Construction 1, each characterized by a constant pair $\{\lambda_1, \lambda_2\}$. However, $\{\lambda_1, \lambda_2\}$ are not arbitrary. They should be chosen to assure the positivity of $M^P(X)$ in the admissible domain of X.[15]

An identity involving the confluent hypergeometric function, $\Phi(a,a;z) = e^z\ \forall a, z$, gives rise to the following two interesting special cases of the general solution (12):

(1) $\lambda_2 = 0, \delta = \gamma$: in this case,

$$M^P(X,t) = e^{-\rho t}e^{-(\alpha+\beta)(H_0+H_1X)}, \tag{13}$$

which is well known in DTSM literature as exponential affine (see e.g., Duffie *et al.* (2000)). In particular, this is a completely affine configuration because the resulting mpr is $\eta^{QP} \sim \sigma^X = \sqrt{H_0 + H_1 X}$, and P-dynamics $\mu^{X,P}$ are affine in X.

(2) $\lambda_1 = 0, \delta = 1$: in this case,

$$M^P(X,t) = e^{-\rho t}e^{-(\alpha+\beta)(H_0+H_1X)}(H_0 + H_1 X)^{\gamma-1}, \tag{14}$$

[15] Because SDFs are determined up to a multiplicative constant, $\{\lambda_1, \lambda_2\}$ can be chosen to satisfy the normalization $M^P(X(0)) = 1$.

which offers new and richer SDFs that also contain a polynomial factor of X (referred to as polynomial–exponential-affine hereafter). Remarkably, the P-dynamics implied by this SDF are also affine, even though mpr η^{QP} associated with M^P does not have to be proportional to the state variable's volatility σ^X. We will analyze and apply this special SDF structure in Secs. 3.4 and 5, to generate the FPP or a negative correlation between changes in exchange rate and interest rate differentials. To get a perspective, FPP is not accommodated by simple exponential affine SDFs (Backus *et al.*, 2001).

3.1.2. *Relations to affine DTSMs*

In our construction, the mpr is implied from (5) and P-dynamics drift from (9)

$$\eta^{QP}(X) = -\frac{M_X^P(X)}{M^P(X)}\sigma^X(X),$$

$$\mu^{X,P}(X) = \mu^{X,Q}(X) + \eta^{QP}(X)\sigma^X(X) = K_0^Q + K_1^Q X - \frac{M_X^P(X)}{M^P(X)}(H_0 + H_1 X).$$

It is clear from the solution (12) that this construction features non-affine P-dynamics $\mu^{X,P}(X)$. In comparison with leading affine DTSMs, where both P- and Q-dynamics are affine, the tradeoff is evident.[16] Our SDF construction features rich (non-affine) P-dynamics at the price of a more restrictive choice of mpr (needed to enforce proper functional SDF in our construction). Before turning to a more general SDF construction, we characterize this tradeoff quantitatively in following proposition.

Proposition 3. *In one-factor settings with affine rfr $r = a + bX$:*

(i) *The functional SDF in Construction 1, together with the following additional specifications:*

$$K_0^P - K_0^Q = 0, \quad \frac{1}{2}[(K_1^P)^2 - (K_1^Q)^2] = b,$$

is a model in either the completely or essentially affine DTSM classes.

(ii) *The functional SDF in Construction 1, together with the following additional specifications:*

$$(K_0^P - K_0^Q)(K_0^P + K_0^Q - 1) = 0, \quad \frac{1}{2}[(K_1^P)^2 - (K_1^Q)^2] = b,$$

is a model in the extended affine DTSM class.

[16] Completely affine models by Dai and Singleton (2000), essentially affine models by Duffee (2002), and extended affine models by Cheridito *et al.* (2007).

3.2. *Construction 2: Equivalent measure R*

A simple observation generalizes our SDF construction substantially. In a nutshell, the introduction of an equivalent measure R (which is not necessarily risk-neutral Q or physical P) to SDF construction can facilitate (i) non-affine Q- and P-dynamics $\mu^{X,Q}(X)$, $\mu^{X,P}(X)$, (ii) non-affine rfr $r(X)$, (iii) non-Markovian (path-dependent) SDFs, while (iv) keeping bond pricing tractable. In the difference with the risk-neutral probability, there is no apparent link, and thus, poses a constraint on R-dynamics $\mu^{X,R}(X)$, directly from price data.

We start by noting that no-arbitrage pricing may be performed in P, Q or any equivalent measure R. The pricing of a contingent payoff $D(X, T)$ in different measures reads

$$E_t^Q\left[\frac{\exp\left(-\int^T r(X,s)\,ds\right)}{\exp\left(-\int^t r(X,s)\,ds\right)}D(X,T)\right] = E_t^R\left[\frac{M^R(X,T)}{M^R(X,t)}D(X,T)\right],$$

where M^R is the SDF associated with equivalent measure R. The tractability of bond pricing is extended to a R-affine framework by employing the Fourier transform $\widehat{M}^R$ of SDF M^R (e.g., Chen and Joslin (2012) and Cuchiero *et al.* (2009)),

$$\begin{aligned}\widehat{M}^R(v,t) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ivX}M^R(X,t)\,dX \leftrightarrow M^R(X,t)\\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ivX}\widehat{M}^R(v,t)\,dv. \end{aligned} \tag{15}$$

Assuming the existence of $\widehat{M}^R(v)$, affine zero-coupon bond pricing in equivalent measure R proceeds as usual,

$$\begin{aligned}\mathrm{ZCB}_{t,t+T} &= E_t^R\left[\frac{M^R(X(t+T),t+T)}{M^R(X(t),t)}\right]\\ &= \frac{e^{-\rho T}}{\sqrt{2\pi}}\frac{1}{M^R(X(t))}\int_{-\infty}^{\infty}\widehat{M}^R(v)E_t^R\left[e^{ivX(t+T)}\right]dv\\ &= \frac{e^{-\rho T}}{\sqrt{2\pi}}\frac{1}{M^R(X(t))}\int_{-\infty}^{\infty}\widehat{M}^R(v)\,e^{A(v,T)+B(v,T)X(t)}\,dv. \end{aligned} \tag{16}$$

Alternatively, when M^R has certain functional forms, e.g., product of polynomial and exponential functions $M^R = e^{\alpha X}X^n$, the bond pricing in measure R proceeds by repeated differentiating (similar to computing moments using the moment generating function)

$$\mathrm{ZCB}_{t,t+T} = E_t^R\left[\frac{M^R(X(t+T),t+T)}{M^R(X(t),t)}\right] \sim \frac{\partial^n}{\partial\alpha^n}\bigg|_{\alpha=0} E_t\left[e^{\alpha X(t+T)}\right]$$
$$= \frac{\partial^n}{\partial\alpha^n}\bigg|_{\alpha=0} e^{A(\alpha,T)+B(\alpha,T)X(t)}.$$

The last equality arises thanks to affine state dynamics in measure R. These flexibilities allow for non-affine rfr, and relate our SDF construction to more general DTSMs such as quadratic, quotient, and other models.

Construction 2. *Let R be an (any) equivalent measure. Assume the following:*

- *SDF is a proper, but unspecified, function of the state variable in measures P and R: $M^P(X,t) = e^{-\rho t}M^P(X)$, $M^R(X,t) = e^{-\rho t}M^R(X)$.*[17] *Furthermore, $M^R(X)$ is a bounded function,*
- *affine R-dynamics: $\mu^{X,R}(X) = K_0^R + K_1^R X$; $(\sigma^X(X))^2 = H_0 + H_1 X$,*
- *Q-dynamics drift $\mu^{X,Q}(X)$ is some given function of the state variable.*[18]

Note that the state dynamics in equivalent measure R are not directly constrained by observed prices. This gives rise to substantial modeling flexibilities to compensate for the functional SDF restriction in Construction 2. The boundedness of $M^R(X)$ is a sufficient condition for the existence of its Fourier transform $\widehat{M}^R(v)$, and the subsequent tractable bond pricing. R-SDF $M^R(X,t)$ is implied in the construction by integrating out the first-order differential equation (see (A.6)),

$$(\sigma^X(X))^2 M_X^R(X) + [\mu^{X,R}(X) - \mu^{X,Q}(X)]M^R(X) = 0$$
$$\Rightarrow M^R(X) = \lambda \exp\left\{-\int^X \frac{\mu^{X,R}(x)-\mu^{X,Q}(x)}{(\sigma^X(x))^2}\,dx\right\}, \qquad (17)$$

where λ is the constant of integration. Subject to the boundedness of $M^R(X,t)$, this is the most general and possible SDF in measure R that is consistent with the given dynamics $\{\mu^{X,Q},\mu^{X,R},\sigma^X\}$. Several observations

[17] It is straightforward to incorporate the more general configurations where rates ρ^P and ρ^R are different. This arises, for example, in models where measure R characterizes the representative agent's subjective belief and the true discount rate ρ^P confounds the belief to produce an effective discount rate ρ^R, e.g., Yan (2008). However, this flexibility does not present a new construction concept, and will be omitted for the sake of simple exposition.

[18] The choice of Q-dynamics $\mu^{X,Q}(X)$ is either dictated by price data, as in Breeden and Litzenberger (1978), or specified as in affine DTSMs of interest rate, and respects the regularity of state variable X.

concerning the SDF (17) are in order here. First, when state dynamics follow the Ornstein–Uhlenbeck (OU) mean-reverting process in both measures Q and R ($H_1 = 0$): $M^R(X) \sim \exp(A + BX + CX^2)$, which has the exponential–quadratic functional form studied by Constantinides (1992). We will study this functional SDF in the next section. Second, when the state dynamics follow the CIR process in both measures ($H_1 \neq 0$): $M^R(X) \sim \exp(A + BX)(H_0 + H_1 X)^C$, which has the polynomial–exponential-affine functional form. We thus reconfirm (14), which was derived under a similar assumption of CIR square-root Q- and P-dynamics. Finally, in Construction 2, a specification for risk-neutral dynamics $\mu^{X,Q}(X)$ is still needed because we do not wish to explicitly specify SDFs $M^P(X)$ and $M^R(X)$. This construction is not covered by Proposition 1 because rfr $r(X)$ is not given here. Construction 3 below provides further generalizations.

3.2.1. *Non-affine interest rate*

A property of Construction 2 is the non-affine dynamics $\mu^{X,P}(X)$, $\mu^{X,Q}(X)$ in both measures P and Q. This is a result of enforcing the tractability of bond pricing (affine dynamics) in equivalent measure R, while setting loose the dynamics in measures Q and P. Consequently, rfr is non-affine in the state variable (see (A.6)),

$$r(X) = \rho + \frac{-[\mu^{X,Q}(X)]^2 - \mu_X^{X,Q}(X) + \mu^{X,Q}(X)H_1 + [K_0^R + K_1^R X]^2 + K_1^R H_0 - K_0^R H_1}{H_0 + H_1 X}. \tag{18}$$

Flexibility in the choice of $\mu^{X,Q}$ translates into the flexible modeling of interest rates. To illustrate, the non-parametric studies of Ait-Sahalia (1996) and Stanton (1997) point to a diffusion term of power $\delta \approx 1.5$ in the rfr process $dr = \mu^r(r)dt + \sigma r^\delta dZ^P$. Meanwhile, one-factor affine dynamic setting with an affine interest rate can only generate either $\delta = 0$ or $\delta = 0.5$. Appropriate specification of $\mu^{X,Q}(X)$ in contrast can give rise to a wider choice for δ in (18).[19]

Note that up to this point in Construction 2, we have not used Assumption 1 on functional SDF M^P in measure P. This assumption will be needed if we wish to pin down $M^P(X,t)$ using the strategy of Construction 1 above. Indeed, Construction 2 implies rfr $r(X)$ as in (18) (whereas $r(X)$ is specified in Construction 1). With Assumption 1 in place, SDF $M^P(X,t)$ in

[19] One might start with some exogenously specified and non-affine $r(X)$ and proceed to P-SDF $M^P(X,t)$ along the line of Proposition 1. Such an approach, however, does not lead to tractable bond pricing in general.

physical measure then follows from linear differential equation (8). We explicitly carry out this procedure next.

3.2.2. *Quadratic DTSMs*

We consider the first special case of Construction 2, wherein the state variable has affine dynamics also in risk-neutral measure, with constant volatility (i.e., OU mean-reverting process): $\mu^{X,Q}(X) = K_0^Q + K_1^Q X$; $[\sigma^X(X)]^2 = H_0$. The interest rate in this model is quadratic in X (18)

$$r(X) = \rho_0 + \rho_1 X + \rho_2 X^2,$$
$$\rho_0 \equiv \rho + \frac{(K_0^R)^2 - (K_0^Q)^2 - K_1^Q - K_1^R H_0}{H_0}; \quad \rho_1 \equiv \frac{2(K_0^R K_1^R - K_0^Q K_1^Q)}{H_0}; \tag{19}$$
$$\rho_2 \equiv \frac{(K_1^R)^2 - (K_1^Q)^2}{H_0}.$$

It is interesting to see that this is the quadratic DTSM developed in Ahn *et al.* (2002) and Leippold and Wu (2002), by specifying OU mean-reverting state dynamics and quadratic rfr.[20] Hence, Construction 2, when specialized to mean-reverting Q-dynamics, relates to this quadratic DTSM framework in the literature. However, since we also assume that P-SDF is a proper function of state variable $M^P(X,t) = e^{-\rho t} M^P(X) = \frac{e^{-\rho t}}{\phi^P(X)}$, we can construct its governing linear differential equation. Substituting quadratic rfr (19) into Proposition 1 yields

$$\frac{1}{2} H_0 \phi_{XX}^P(X) + (K_0^Q + K_1^Q X)\phi_X^P(X) + (\rho - \rho_0 - \rho_1 X - \rho_2 X^2)\phi^P(X) = 0. \tag{20}$$

This equation determines all possible SDF functions $M^P(X,t)$.

Proposition 4. *The most general functional SDF M^P consistent with Construction 2, when the latter is specialized to OU mean-reverting Q-dynamics (i.e., quadratic DTSM), is*

$$M^{P,\{\lambda_1,\lambda_2\}}(X,t) = \frac{e^{-\rho t} e^{-MX-NX^2}}{\lambda_1 e^{\frac{-(m+nX)^2}{4}} \mathbf{\Phi}\left(\frac{\nu}{2}+\frac{1}{4}, \frac{1}{2}; \frac{(m+nX)^2}{2}\right) + \lambda_2 (m+nX) e^{\frac{-(m+nX)^2}{4}} \mathbf{\Phi}\left(\frac{\nu}{2}+\frac{3}{4}, \frac{3}{2}; \frac{(m+nX)^2}{2}\right)}, \tag{21}$$

where λ_1, λ_2 are constants of integration.

[20] These quadratic DTSMs specify OU state dynamics in measure P, and impose an affine mpr. These together imply OU dynamics in measure Q, which facilitates a tractable risk-neutral pricing of bonds, given a quadratic $r(X)$.

Appendix A gives a proof of this proposition, together with expressions for coefficients ν, m, n, M, N. z is linear in X, and $\mathbf{\Phi}(.,.;z)$ is a confluent hypergeometric function of argument z. Two particularly simple cases arise when ν assumes special values.[21]

(1) $\nu = \frac{3}{2}$, $\lambda_1 = 0$: in this case,

$$M^P(X,t) = e^{-\rho t}\frac{1}{m+nX}\exp\biggl(-MX - NX^2 - \frac{1}{4}(m+nX)^2\biggr).$$

The resulting P-dynamics $\mu^P(X,t)$ (9) are non-affine $\frac{A}{X} + B + CX$.

(2) $\nu = \frac{1}{2}$, $\lambda_2 = 0$: in this case,

$$M^P(X,t) = e^{-\rho t}\exp\biggl(-MX - NX^2 - \frac{1}{4}(m+nX)^2\biggr),$$

is an exponential–quadratic SDF under physical measure P. The resulting P-dynamics are affine (just as the given Q and R-dynamics in the current setting). This is a strong reminiscence of the SAINTS model introduced in Constantinides (1992),[22] which is analyzed in connection with Construction 3 below.

For other general values of ν and λ_1, λ_2, R-SDF $M^R(X,t)$ follows (17), which is an exponential quadratic function of the state variable in the current quadratic setting. In light of Proposition 4, we can start out with either affine R-dynamics (then coefficients $\delta_0, \delta_1, \delta_2$ are related to the construction's original parameters via (19)), or a quadratic rfr $r(X) = \rho_0 + \rho_1 X + \rho_2 X^2$ (then $\delta_0, \delta_1, \delta_2$ are exogenous, and only subject to the positivity of $r(X)$ in admissible domain of X). In comparison with quadratic DTSMs of Ahn *et al.* (2002) and Leippold and Wu (2002), our construction has rich P-dynamics (non-affine drift $\mu^P(X,t)$ and mpr η^{QP}) at the price of a more restrictive (functional) SDF $M(X,t)$. Both approaches give rise to quadratic forward rates and equally tractable bond pricing.

3.2.3. *Quotient DTSM*

In another special case of Construction 2, Q-dynamics are also affine (a square-root process) $\mu^{X,Q}(X) = K_0^Q + K_1^Q X$, $H_0 = 0$. A quotient rfr

[21] This amounts to imposing a constraint on model's parameters. See Appendix A for the expression of ν. The identity $\Phi(a,a;z) = e^z \ \forall a, z$ of confluent hypergeometric function is behind these results.

[22] SAINTS stands for "Squared autoregressive independent state variable nominal term structure."

process then follows from Eq. (18)

$$r(X) = \theta_{-1}\frac{1}{X} + \theta_0 + \theta_1 X,$$
$$\theta_{-1} \equiv K_0^Q - K_0^R + \frac{(K_0^R)^2 - (K_0^Q)^2 - K_1^Q}{H_1}; \tag{22}$$
$$\theta_0 \equiv \rho + \frac{2(K_0^R K_1^R - K_0^Q K_1^Q)}{H_1} + K_1^Q; \quad \theta_1 \equiv \frac{(K_1^R)^2 - (K_1^Q)^2}{H_1}.$$

General formula (17) then yields SDF in equivalent measure R

$$M^R(X,t) = e^{-\rho t} X^{(K_0^Q - K_0^R)/H_1} \exp\left(\frac{K_1^Q - K_1^R}{H_1} X\right). \tag{23}$$

Not surprisingly, $M^R(X,t)$ has the same polynomial–exponential-affine form as $M^P(X,t)$ in (14), because this is the most general functional form of SDF, consistent with square-root dynamics in respective measure pairs (R and Q, or P and Q, as discussed below (17)). However, the approaches to these functional SDFs are quite different. For a comparison, we note that measure P of the setting leading to (14) has a similar role to measure R of the current setting. The current rfr (22) is implied and more general than the affine function $r(X) = a + bX$ of Construction 1. Consequently, in physical measure P, the current SDF $M^P(X,t) \equiv e^{-\rho t}\frac{1}{\phi^P(X)}$ is determined by a special version of different equation (8)

$$\frac{1}{2}H_1 X \phi^P_{XX}(X) + (K_0^Q + K_1^Q X)\phi^P_X(X) + \left(\rho - \theta_{-1}\frac{1}{X} - \theta_0 - \theta_1 X\right)\phi^P(X) = 0, \tag{24}$$

The fact that $M^P(X,t)$ is different from (14) of previous Construction 1 is confirmed by the following result.

Proposition 5. *The most general functional SDF M^P consistent with Construction 2, when the latter is specialized to CIR square-root Q-dynamics (or equivalently, quotient rfr), is*

$$M^{P,\{\lambda_1,\lambda_2\}}(X,t) = e^{-\rho t}\frac{e^{-\alpha X} X^{-\beta}}{\lambda_1 \Phi(\delta,\gamma;z) + \lambda_2 z^{1-\gamma}\Phi(\delta - \gamma + 1, 2 - \gamma; z)},$$
$$z \equiv \frac{2[(K_1^Q)^2 + 2H_1\theta_1]^{1/2}}{H_1} X, \tag{25}$$

where λ_1, λ_2 are constants of integration.

Appendix A gives a proof of this proposition, together with expressions for coefficients $\alpha, \beta, \gamma, \delta$. z is linear in X, and $\mathbf{\Phi}(.,.;z)$ is a confluent hypergeometric function of argument z. From (9) then follow mpr $\eta^{QP} = \frac{-\sigma^X M_X^P}{M^P}$ and P-dynamics $\mu^{X,P} = \mu^{X,Q} + \sigma^X \eta^{QP}$, which are generally non-affine. Bond pricing is tractable by (16), or other transforms of Duffie *et al.* (2000). We note that Proposition 5 holds regardless of whether we begin with the given quotient rfr or affine dynamics in some equivalent R. In the latter case, coefficients $\theta_{-1}, \theta_0, \theta_1$ are given in (22).

3.3. *Construction* 3: *Bypassing risk-neutral measure*

All constructions so far have required the specification of Q-dynamics $\{\mu^{X,Q}, \sigma^{X,Q}\}$. This is because pricing in interest rate models is most conveniently implemented in risk-neutral measure, given a rfr. In this framework, state variables' Q-dynamics are specified to produce closed-form bond prices. Alternatively, Q-dynamics may also be inferred from asset prices (Breeden and Litzenberger, 1978). However, as the previous section shows, not only can pricing be done in equivalent measure (16), but also rfr r can be implied (not specified) from a functional SDF assumption. This suggests generalizations of SDF construction by replacing the specification of risk-neutral dynamics with that of equivalent measure. As a special case, these generalizations obtain the class of SAINTS models of Constantinides (1992).

Construction 3. *Let R be an* (*any*) *equivalent measure. Assume the following*:

- *SDF is a proper function of the state variable in measures P and R*: $M^P(X,t) = e^{-\rho t} M^P(X)$, $M^R(X,t) = e^{-\rho t} M^R(X)$. *Function $M^R(X)$ is exogenously specified and bounded, whereas $M^P(X)$ is implied*,
- *affine R-dynamics*: $\mu^{X,R}(X) = K_0^R + K_1^R X$; $(\sigma^X(X))^2 = H_0 + H_1 X$.

This construction clearly does not rely on any risk-neutral specification. Similar to Construction 2, Fourier transform of the bounded function $M^R(X,t)$ exists to give rise to tractable bond pricing (16). The specification of SDF $M^R(X,t)$ in equivalent measure replaces the rfr specification $r(X,t)$ in Construction 1.[23] The advantage of Construction 3 is in the flexible choice of equivalent measure R. Assumption 1 on functional P-SDF helps to determine M^P consistently. However, because both r and $\mu^{X,Q}$ are not specified, we first need to establish a version of Eq. (8) for the Construction 3. We

[23] In measure Q, $\exp(-\int^t r(X,s)ds)$ is the pricing kernel (15).

note that the change between measures P and R is implemented by the Radon–Nikodym derivative $\xi^{RP} = \frac{M^P}{M^R}$ which is a P-martingale, $d\xi^{RP}(X,t) = -\xi^{RP}(X,t)\eta^{RP}(X)dZ^P(t)$, where $\eta^{RP}(X) = \frac{\mu^{X,P}(X)-\mu^{X,R}(X)}{\sigma^X(X)}$. Similar to the derivation of Eq. (8), by combining Itô's lemma with the change of variable $M^P(X,t) = e^{-\rho t}\frac{1}{\phi^P(X)}$ in (7), we obtain a differential equation

$$\frac{[\sigma^X(X)]^2}{2}\phi^P_{XX}(X) + \left(\mu^{X,R}(X) + \frac{[\sigma^X(X)]^2 M^R_X(X)}{M^R(X)}\right)\phi^P_X(X) + \left(\frac{\mu^{X,R}(X)M^R_X(X)}{M^R(X)} + \frac{[\sigma^X(X)]^2 M^R_{XX}(X)}{2M^R(X)} + \frac{[\sigma^X(X)]^2[M^R_X(X)]^2}{2[M^R(X)]^2}\right)\phi^P(X) = 0. \tag{26}$$

Since $M^R(X,t)$ and R-dynamics $\{\mu^{X,R}(X), \sigma^X(X)\}$ are all specified in Construction 3, this differential equation determines all possible P-SDF $M^P(X,t)$ that are consistent with the construction. Then follow all quantities of interest: interest rate r and R-mpr η^{RQ} from the drift and volatility of M^R, respectively, Q-dynamics $\mu^{X,Q} = \mu^{X,R} - \eta^{QR}\sigma^X$, P-dynamics $\mu^{X,P} = \mu^{X,R} + \eta^{RP}\sigma^X$, and finally P-mpr η^{QP} from the volatility of the above M^P.[24] The implied interest rate and state dynamics in measures P and Q are rich (non-affine). For an illustration of Construction 3, we now consider the SAINTS models.

3.3.1. *SAINTS models in equivalent measure*

The SAINTS models of Constantinides (1992) fit into our SDF construction framework directly because they are built on a functional SDF specification. This class of models is constructed originally in physical measure P with the following ingredients:

(i) OU mean-reverting P-dynamics: $\mu^{X,P} = K^P_0 + K^P_1 X$, $(\sigma^X)^2 = H_0 =$ constant,
(ii) exponential–quadratic P-SDF: $M^P = e^{-\rho t}e^{(X-a)^2}$.

The zero-coupon bond price is tractable because state variable X is a (conditional) Gaussian, hence M^P is a log χ^2 process. The differential representation $\frac{dM^P}{M^P} = -rdt - \eta^{QP}dZ^P(t)$ implies a quadratic rfr $r(X)$, and affine mpr $\eta^{QP}(X) = 2\sqrt{H_0}(a - X)$. The latter in turn implies that Q-dynamics are also affine, $\mu^{X,Q}(X) = \mu^{X,P}(X) - \sigma^X\eta^{QP}(X) = K^P_0 - 2aH_0 + (K^P_1 + 2H_0)X$. We observe that the featuring exponential–quadratic SDF in SAINTS models can also be established the other way around. Once a functional

[24] Note that $\frac{dM^R}{M^R} = -rdt - \eta^{QR}dZ^R(t)$, where $dZ^Q(t) = dZ^R(t) + \eta^{QR}dt$ and $\eta^{QR} = \frac{\mu^{X,R}-\mu^{X,Q}}{\sigma^X}$.

(but unspecified) SDF $M^P(X,t)$ is assumed (Assumption 1), the mean-reverting Q-dynamics unambiguously imply an exponential–quadratic SDF, as we explain in the paragraph next to Eq. (17) above. Therefore, the two approaches, one specifying SDF $M^P(X,t)$ (as in Constantinides (1992), and Construction 3 more generally), and the other specifying risk-neutral dynamics $\mu^{X,Q}(X)$ (as in Construction 2), both lead to SAINTS models.

The introduction and modeling of an equivalent measure in Construction 3, generalizes the SAINTS models in a simple way. We specify SAINTS dynamics in an equivalent measure R (instead of physical P). As a result, bond pricing is tractable and physical dynamics $\mu^{X,P}(X)$ and SDF $M^P(X,t)$ are richer (non-affine). We consider the following specification in line with Construction 3:

(i) OU mean-reverting R-dynamics:s $\{\mu^{X,R} = K_0^R + K_1^R X; (\sigma^X)^2 = H_0\}$, and
(ii) specified R-SDF $M^R = e^{-\rho t} e^{(X-a)^2}$, and functional (but unspecified) P-SDF $M^P(X,t)$.

In this specification, by the substitution of P by R, the pricing of a zero-coupon bond is tractable, rfr $r(X,t)$ is quadratic in X, and the resulting forward rate dynamics are equally analytical, as in the original SAINTS setting.[25] Assumption 1 $M^P(X,t)$ determines M^P via a differential equation (which is a special case of Eq. (26) adapted to an exponential–quadratic $M^R(X,t)$). Not surprisingly, because rfr $r(X)$ is quadratic in the state variable, this second-order linear differential equation has a form identical to (20). An application of Proposition 4 then yields P-SDF in our extended SAINTS framework

$$M^P(X,t) = \frac{e^{-\rho t} e^{-MX-NX^2}}{\lambda_1 e^{\frac{-(m+nX)^2}{4}} \mathbf{\Phi}\left(\frac{\nu}{2}+\frac{1}{4}, \frac{1}{2}; \frac{(m+nX)^2}{2}\right) + \lambda_2 (m+nX) \times e^{\frac{-(m+nX)^2}{4}} \mathbf{\Phi}\left(\frac{\nu}{2}+\frac{3}{4}, \frac{3}{2}; \frac{(m+nX)^2}{2}\right)}, \tag{27}$$

where λ_1, λ_2 are constants of integration. Appendix A gives a derivation of (27), together with expressions for coefficients ν, m, n, M, N. Evidently, SDF $M^P(X,t)$ (27) is more general than an exponential–quadratic function of the original SAINTS models as the associated mpr $\eta^{QP}(X)$ and physical state dynamics $\mu^{X,P}(X)$ are non-affine in our extension.[26]

[25] The pricing of zero-coupon bond is $\mathrm{ZCB}_{t,t+T} = E_t^P[\frac{M^P(X(t+T),t+T)}{M^P(X(t),t)}] = E_t^R[\frac{M^R(X(t+T),t+T)}{M^R(X(t),t)}]$.
[26] $M^P(X,t)$ in (27) becomes an exponential–quadratic function in the special case where $\nu = \frac{1}{2}$ and $\lambda_2 = 0$ (as discussed below Proposition 4), or when equivalent measure R simply coincides the physical P.

3.4. *Summary*

We now briefly summarize the main connections between the current paper's functional SDF approach and the most popular DTSMs in literature. When state dynamics follow OU mean-reverting processes under risk-neutral measure Q and another equivalent measure R (which can be physical measure P as a special case), $(\sigma^X)^2 = H_0 = \text{constant}$, the functional R-SDF is a exponential-quadratic function $M^R(X,t) = e^{-\rho t} e^{A+BX+CX^2}$. The resulting rfr is a quadratic function of the state variable $r(X) = \rho_0 + \rho_1 X + \rho_1 X^2$. Various special cases of this result are covered in (17), (19) and the SAINTS models.

When state dynamics follow CIR square-root processes under risk-neutral measure Q and another equivalent measure R (which can be P as a special case), $(\sigma^X)^2 = H_0 + H_1 X$, the functional R-SDF is a polynomial–exponential-affine function

$$M^R(X,t) = e^{-\rho t} e^{A+BX} (H_0 + H_1 X)^C. \tag{28}$$

The resulting rfr is a quotient function of state variable $r(X) = \theta_{-1} X^{-1} + \theta_0 + \theta_1 X$ in general.[27] Various special cases of this result are covered in (13), (14), (17) and (23).[28] In particular, polynomial–exponential-affine function is the most general functional SDFs that are compatible with the complete-affine DTSMs of Dai and Singleton (2000). This function is more general than the exponential-affine SDF widely considered in literature. This is an important generalization of affine dynamic models to address the FPP in international finance (Sec. 5).

4. LG Dynamics and Beyond

This section studies the functional SDF approach in conjunction with the LG processes of Gabaix (2009). Asset pricing models based on LG processes yield tractable prices for both bonds and stocks (see also Menzly *et al.* (2004)). This tractability is useful in demonstrating how time-varying rare disasters can explain regularities observed in finance and macroeconomics (Gabaix, 2012). Interestingly, LG asset pricing models feature functional SDFs (in fact, linear function of state variables). It is hence directly related to the current paper's functional SDF construction (Assumption 1). We first summarize the original

[27] In special cases, when the model's parameters satisfy certain restrictions, the quotient rfr reduces to an affine function, as in the affine DTSM underlying Eq. (13).

[28] Indeed, as SDFs are determined only up to a multiplicative constant, (14) and (28) are essentially the same, while (13) is a special case of (28) with $C = 0$.

LG framework, before generalizing it in light of the functional SDF construction.

4.1. *LG dynamics and infinitesimal generator*

LG bond pricing models comprise of (i) underlying LG (vector) process $X(t)$ in physical measure defined by $E_t^P[dX(t)] = -\Omega X(t)dt$, where Ω is a (generator) matrix, and (ii) SDF as linear in X: $M^P(X) = \lambda^m \cdot X$, where λ^m is a constant vector.[29] For stock pricing, an additional specification $M^P D(X) = \lambda^{md} \cdot X$ is assumed, where $M^P D$ is the product of P-SDF M^P and dividend process D.

Key LG pricing results can be reproduced by employing the infinitesimal generator $\mathcal{D}^{X,P}$ associated with diffusion process $X(t)$ (1) in physical measure P. For a smooth function $f(X,t)$, $\mathcal{D}^{X,P}f$ presents the drift of the infinitesimal process $df(X,t)$

$$\begin{aligned}\mathcal{D}^{X,P}f(X,t) &\equiv \lim_{\Delta t\to 0}\frac{E_t[f(X(t+\Delta t),(t+\Delta t)) - f(X(t),t)]}{\Delta t}\\ &= f_t(X,t) + \mu^{X,P}(X,t)f_X(X,t)\\ &\quad + \frac{1}{2}\mathrm{Tr}[\sigma^X(X,t)\sigma^{X,T}(X,t)f_{XX}(X,t)],\end{aligned}$$

where Tr and superscript T denote the trace and transpose operator, respectively. When f is the state variable, $f(t,X) = X$, the infinitesimal generator produces $\mathcal{D}^{X,P}X = \mu^{X,P}(X,t)$ and the defining property of LG pricing framework

$$\begin{aligned}&\mathcal{D}^{X,P}X(t) = \mu^{X,P}(X,t) = -\Omega X(t); \quad M^P(X) = \lambda^m \cdot X;\\ &M^P D(X) = \lambda^{md}\cdot X.\end{aligned} \tag{29}$$

By induction, we have $E_t^P[X(t+T)] = e^{-T\Omega}X(t)$, and the price of the zero-coupon bond,[30]

$$\mathrm{ZCB}(t,X) = E_t^P\left[\frac{M^P(X,t+T)}{M^P(X,t)}\right] = \frac{\lambda^m \cdot E_t^P[X(t+T)]}{\lambda^m\cdot X(t)} = \frac{\lambda^m\cdot e^{-T\Omega}X(t)}{\lambda_m\cdot X(t)}.$$

[29] Both Ω and λ^m can vary with time t, but are independent of state variable X, to assure tractable asset prices of the model.

[30] By induction, $E_t^P[X(t+T)] = X(t) + E_t^P[\int_t^{t+T} dX(s)] = X_t - \int_t^{t+T}\Omega E_t^P[X(s)]ds = X_t - T\Omega X(t) + \Omega^2\int_t^{t+T}\int_t^s E_t^P[X(\tau)]d\tau ds = \cdots = e^{-T\Omega}X(t)$, wherein the matrix-exponential operation is the limit of the usual Taylor expansion, $e^{-T\Omega} = \sum_{n=0}\frac{(-T\Omega)^n}{n!}$. A sufficient condition for this convergence is that all eigenvalues of Ω are strictly positive.

Similarly, stock contingent on dividend stream $\{D(X,t)\}$ also possesses closed-form price[31] (assuming usual regularity conditions to interchange the order of integration and expectation operations)

$$P(X,t) = E_t^P\left[\int_t^{t+T} \frac{M^P(X,s)D(X,s)}{M^P(X,t)}\,ds\right] = \frac{\lambda^{md}\cdot\int_t^{t+T} E_t^P[X(s)]\,ds}{M^P(X,t)}$$
$$= \frac{\lambda^{md}\cdot\left[\int_t^{t+T} e^{-(s-t)\Omega}ds\right]X(t)}{M^P(X,t)}.$$

4.1.1. *LG pricing models vs. affine DTSMs*

Apparently, linear drift $\mu^{X,P} = -\Omega X(t)$ of LG processes is a strong reminiscence of affine DTSMs. However, the resemblance stops here. First, the rfr in LG pricing models can be determined from the differential representation $\frac{dM^P}{M^P} = -rdt - \eta^{QP}dZ^P(t)$

$$r(X) = -\frac{\mathcal{D}^{X,P}M^P(X,t)}{M^P(X,t)} = -\frac{\lambda_m\cdot\mathcal{D}^{X,P}X(t)}{\lambda_m\cdot X(t)} = \frac{\lambda_m\Omega X(t)}{\lambda_m\cdot X(t)}. \tag{30}$$

Interest rates in the LG framework are rational in state variable X, but not affine as in affine DTSMs.

Second, as far as bond pricing is concerned, LG models do not place outright restrictions on the state variable's diffusion $(\sigma^X)^2$, whereas affine DTSMs impose a linear structure $(\sigma^X)^2 = H_0 + H_1X$. However, LG models also aim to price stocks analytically, and hence, implicitly generate a certain specification. This specification follows directly from the LG defining properties $\mathcal{D}^{X,P}[M^PD(X,t)] = \mathcal{D}^{X,P}[-\lambda^{md}\cdot X] = -\lambda^{md}\cdot\Omega X(t)$ (29),

$$\begin{aligned}
&-\lambda^{md}\cdot\Omega X(t)\\
&\quad= -\lambda^m\cdot\Omega X(t)D(X,t) + \frac{1}{2}M^P(X,t)\mathrm{Tr}(\sigma^X(X,t)\sigma^{X,T}(X,t)D_{XX}(X,t))\\
&\qquad+ M^P(X,t)\left[-\Omega X(t) + \sigma^X(X,t)\sigma^{X,T}(X,t)\frac{M_X^P(X,t)}{M^P(X,t)}\right]\cdot D_X(X,t).
\end{aligned}$$

Substituting explicit relationships $M^P = \lambda^m\cdot X(t)$, $D = \frac{MD}{M} = \frac{\lambda^{md}\cdot X(t)}{\lambda^m\cdot X(t)}$ of (29) into the above equation produces a constraint on the diffusion σ^X of the underlying LG process.[32] In sum, the affine DTSMs and LG approaches to tractable bond pricing are different. While canonical DTSMs specify affine

[31] Explicit bond and stock pricing requires the additional step of diagonalizing the generator Ω to implement the exponential matrix operation $e^{-T\Omega}$.

[32] In settings with vector state variables, this constraint is not sufficient to pin down matrix $\sigma^X\sigma^{X,T}$ unambiguously.

rfr, affine Q-dynamics, and imply non-linear SDF, LG pricing models specify linear SDF $M^P(X)$, affine drift on P-dynamics, and imply non-affine rfr. As a result, affine DTSMs generate a linear forward rate, while LG models give rise to closed-form stock pricing.

4.2. *Extension to LG modeling*

We now analyze the connection between LG and functional SDF approaches, and the generalizations to the former.

4.2.1. *LG pricing models vs. functional SDF construction*

LG pricing models possess a functional SDF, and center around the eigenproblem of infinitesimal generator: $\mathcal{D}^P X = -\Omega X$ (29). From martingale pricing perspectives, Radon–Nikodym derivative $\xi^{QP} \equiv e^{\int^t r(X,s)\,ds} M^P(X,t)$ is P-martingale and $\xi^{PQ} \equiv e^{-\int^t r(X,s)\,ds} \frac{1}{M^P(X,t)}$ is Q-martingale. They have null drifts under respective measures, $\mathcal{D}^{X,P}\xi^{QP} = 0$, $\mathcal{D}^{X,Q}\xi^{PQ} = 0$. In particular, the latter implies both differential equation (8) for the functional SDF construction, and the rfr (30) for LG models (after the substitution $M^P(X,t) = \lambda^m \cdot X$). Therefore, our functional SDF construction specifies $r(X,t)$ and lets loose $M^P(X,t)$, where LG model specifies $M^P(X,t)$ and lets loose $r(X,t)$.

Before extending the original LG framework, we observe that LG specifications are invariant with respect to measures (see (A.9)). Therefore, introducing an equivalent measure (along Construction 3) does not generalize the LG setting.

4.2.2. *Extending LG framework*

The extension starts out with non-LG, thus more general, vector dynamics

$$dX(t) = \mu^{X,P}(X,t)\,dt + \sigma^X(X,t)\,dZ^P(t),$$

where $\mu^{X,P}(X,t)$ is not necessarily linear in X. This extension is, therefore, most handy in settings where the state variable dynamics are given (e.g., to meet empirical constraints or pricing/statistical aspects of the model). We construct a new state variable vector $Y = F(X,t)$ (as function of original X) that has desirable LG dynamics, $\mathcal{D}^{X,P}F(X,t) = -\omega F(X,t)$. That is, for all components F^i of vector F

$$\begin{aligned}&\frac{1}{2}\mathrm{Tr}\big[\sigma^X(X,t)\sigma^{X,T}(X,t)F^i_{XX}(X,t)\big] + \mu^{X,P}(X,t)\cdot F^i_X(X,t) + F^i_t(X,t)\\ &\quad = -\sum_j \Omega^{ij}F^j(X,t),\end{aligned} \tag{31}$$

where F_X^i and F_{XX}^i denote, respectively, gradient vector and Hessian matrix of scalar component F^i. Finally, we specify a linear P-SDF, $M^P = \lambda^m \cdot Y$, for tractable bond pricing (and $M^P D = \lambda^{md} \cdot Y$ for tractable stock pricing). In essence, (31) constructs a functional SDF from the general dynamics of underlying state variable X, which fits our SDF construction approach. In the special case of original LG bond pricing models, $F(X) = X$ (so $F_{XX} = 0$), which illustrates the irrelevance of volatility specification $\sigma^X(X,t)$ therein.

In practice, given the state dynamics $\{\mu^{X,P}(X,t), \sigma^X(X,t)\}$ in physical measure and a solution Y of Eq. (31), any function of the form $M^P = \lambda^m \cdot Y$, subject to non-negativity and other regularity conditions, is a consistent P-SDF of a tractable bond pricing model

$$\mathrm{ZCB}_{t,t+T} = E_t^P\left[\frac{M^P(X(t+T), t+T)}{M^P(X(t),t)}\right] = \frac{\lambda^m \cdot e^{-T\Omega} F(X,t)}{\lambda^m \cdot F(X,t)},$$

even though the $X(t)$ is not a LG process. Note that the set of dynamics $\{\mu^{X,P}(X,t), \sigma^X(X,t)\}$, that offer a closed-form solution $F(X,t)$ (31), is larger than the linear span of X. This observation underlies the extension of the original LG framework by (31).

We illustrate this extension in a simple example of two independent factors $X = (X_1, X_2)^T$ which follow a $\frac{3}{2}$-power process of Ahn and Gao (1999).[33]

$$\begin{aligned} dX_1(t) &= X_1(t)[a_1 - X_1(t)]\,dt + \sqrt{2}[X_1(t)]^{3/2}\,dZ_1^P(t), \\ dX_2(t) &= X_2(t)[a_2 - X_2(t)]\,dt + \sqrt{2}[X_2(t)]^{3/2}\,dZ_2^P(t). \end{aligned} \tag{32}$$

We note that a closed-form general solution of (31), for the dynamics (32), exists. For illustration, we are content with the following simple and special solution of the transformed state variable $Y(t)$,[34]

$$Y(t) = \begin{pmatrix} Y_1(t) \\ Y_2(t) \end{pmatrix} = e^{-\rho t}\begin{pmatrix} X_1(t)[a_1 + X_1(t)] \\ X_2(t)[a_2 + X_2(t)] \end{pmatrix}; \quad \Omega = \begin{pmatrix} \rho - a_1 & 0 \\ 0 & \rho - a_2 \end{pmatrix}.$$

Evidently, $Y(t)$ (together with generator Ω) follows LG dynamics. This in turn suggests a simple functional SDF,

$$M^P(X,t) = e^{-\rho t}\{\lambda^{m,1} X_1(t)[a_1 + X_1(t)] + \lambda^{m,2} X_2(t)[a_2 + X_2(t)]\},$$

[33] This process evidently does not belong to the original LG class.

[34] Any solution, regardless of how special, is consistent with same state X dynamics (32) and has a similar LG pricing power by construction. In practice, this feature renders both flexibility and ease to incorporate extension to LG modeling.

where $\lambda^{m,1}$, $\lambda^{m,2}$ are two constants. This SDF then produces closed-form zero-coupon bond prices

$$\mathrm{ZCB}_{t,t+T} = \frac{\lambda^{m,1} e^{-T(\rho - a_1)} X_1(t)[a_1 + X_1(t)] + \lambda^{m,2} e^{-T(\rho - a_2)} X_2(t)[a_2 + X_2(t)]}{\lambda^{m,1} X_1(t)[a_1 + X_1(t)] + \lambda^{m,2} X_2(t)[a_2 + X_2(t)]}.$$

5. Application: The FPP

This section explores an application of our functional SDF construction to address a pricing anomaly in international finance, namely the FPP. It is the empirical regularity that, with respect to a base currency and the associated nominal interest rate, other currencies tend to appreciate when their nominal interest rates also increase (Fama (1984); see (34) below). This pattern is also commonly referred to as a violation of the uncovered interest rate parity. We first quantify FPP in relation with SDFs, following the approach of Backus *et al.* (2001). We then present a risk-based model featuring functional SDFs that accommodates this puzzle.

5.1. *FPP and affine dynamics*

To set the notation, superscripts h and f denote quantities pertaining to home and foreign countries, respectively. Let the exchange rate $S(t)$ be units of the home currency that buys one unit of the foreign currency at t. Assuming complete international financial markets, that each country has a unique SDF, and the exchange rate equals to the ratio of SDFs of the two countries involved,

$$S(t) = \frac{M^{f,P}(t)}{M^{h,P}(t)} \quad \text{or} \quad s(t) = m^{f,P}(t) - m^{h,P}(t), \tag{33}$$

where lower-case letters denote logarithms of appropriate quantities. In light of this complete-market relationship, the FPP pattern that the foreign currency tends to appreciate when its rfr drops, relatively translates into a negative covariance in physical measure Backus *et al.* (2001),[35]

$$\begin{aligned}\sigma^{ds,\Delta r} &\equiv \mathrm{Cov}(E_t[s(t+dt)] - s(t), r^h(t) - r^f(t)) \\ &= \mathrm{Cov}\left(r^h(t) + \frac{[\eta^{QP,h}]^2}{2} - r^f(t) - \frac{[\eta^{QP,f}]^2}{2}, r^h(t) - r^f(t)\right) < 0, \end{aligned} \tag{34}$$

[35] Up to a multiplicative constant, the covariance $\sigma^{ds,\Delta r}$ (34) is the coefficient a_2 defined in Eq. (2) of Backus *et al.* (2001), who also relate it to the slope coefficient of Fama (1984)'s forward premium regression.

where the last equality arises from an application of Itô's lemma on (33), and $\eta^{QP,i}$ is country i's mpr (2). Intuitively, the negative sign of (34) signifies the FPP, i.e., investors appear to demand a lower premium (i.e., interest rate) for holding a depreciating currency.

Within affine DTSM framework, Backus *et al.* (2001) find that it is difficult, both theoretically and empirically, to accommodate FPP (34). To motivate a risk-based model of the next section, it is instructive to briefly reproduce their arguments here. Backus *et al.* (2001) first consider a symmetric, independent, and completely affine dynamics for a two-country setting (one independent factor for each country), together with nominal rfr affine in the respective state variables,[36]

$$\begin{cases} dX^i(t) = (K_0^{i,Q} + K_1^{i,Q} X^i)dt + \sqrt{H_0^i + H_1^i X^i}\, dZ^{i,Q}(t); \\ r^i(X) = a^i + b^i X(t); \quad E_t[dZ^{h,Q}(t)dZ^{f,Q}(t)] = 0; \end{cases} \quad i \in \{h, f\}. \tag{35}$$

As in completely affine DTSMs (Dai and Singleton, 2000), mpr is proportional to the state variable's volatility, $\eta^{QP,i}(X^i, t) \sim \sigma^{X^i} = \sqrt{H_0^i + H_1^i X^i}$, for $i \in \{h, f\}$, so that state dynamics is also affine in physical measure P (13). In this setting, the condition (34) becomes

$$\begin{aligned} \sigma^{ds,\Delta r} &= \mathrm{Var}(r^h(t)) + \frac{1}{2}\mathrm{Cov}([\eta^{QP,h}]^2, r^h(t)) + (h \leftrightarrow f) \\ &= (b^h)^2 \mathrm{Var}(X^h(t)) + \frac{1}{2} H_1^h b^h \mathrm{Var}(X^h(t)) + (h \leftrightarrow f) < 0, \end{aligned} \tag{36}$$

where $(h \leftrightarrow f)$ denotes the repetition of terms, but with the home quantities replaced by the foreign counterparts. First, for either countries, the admissible domain $H_0^i + H_1^i X^i > 0$ for positive, possibly unbounded, square-root process $X^i(t)$ implies that $H_1^i > 0$. Second and by similar reason, almost surely positive rfr $a^i + b^i X^i > 0$ requires $b^i > 0$, $\forall i \in \{h, f\}$. These two observations render the inequality (36) impossible, and consequently, the FPP is inconsistent with the given affine dynamics. Backus *et al.* (2001) then relaxes assumptions on single and independent factors, and still finds that these generalizations fare poorly in addressing the puzzle empirically. Our model below features generalizations to market prices of risk and functional SDFs to accommodate FPP.

[36] Note that due to cancellation, symmetric factors common to both countries do not contribute to covariance (34).

5.2. *FPP-consistent models*

Our FPP-consistent model is based on Construction 1 in Sec. 3.1. We consider a functional SDF of the polynomial–exponential-affine form (28) for each country in physical measure P, and the resulting mpr,

$$M^{i,P}(X^i,t) = e^{-\rho^i t} e^{A^i + B^i X^i}(H_1^i X^i)^{C^i},$$
$$\Rightarrow \eta^{i,QP}(X^i) = -B^i\sqrt{H_1^i X^i} - \frac{C^i H_1^i}{\sqrt{H_1^i X^i}}, \quad i \in \{h,f\}, \tag{37}$$

where we have chosen $H_0^i = 0, \forall i \in \{h,f\}$ for simplicity. We otherwise retain the specification (35) of Backus *et al.* (2001), namely (i) affine, almost surely positive rfr and (ii) independent, symmetric Q-affine dynamics for each country. The polynomial–exponential-affine SDF (37) implies affine dynamics in physical measure

$$dX^i(t) = (K_0^{i,P} + K_1^{i,P} X^i)dt + \sqrt{H_1^i X^i}\, dZ^{i,P}(t),$$
$$K_0^{i,P} = K_0^{i,Q} - C^i H_1^i, \quad K_1^{i,P} = K_1^{i,Q} - B^i H_1^i, \quad i \in \{h,f\}.$$

In summary, our construction is specified by (35) and (37). In our model, the necessary condition (34) for FPP becomes

$$\sigma^{ds,\Delta r} = (b^h)^2 \mathrm{Var}(X^h(t)) + \frac{1}{2}(B^h)^2 H_1^h b^h \mathrm{Var}(X^h(t))$$
$$+ \frac{1}{2}(C^h)^2 H_1^h b^h \mathrm{Cov}\left(X^h(t), \frac{1}{X^h(t)}\right) + (h \leftrightarrow f) < 0, \tag{38}$$

where b's are from (35), and B's, C's, and H_1's are from (37). A comparison of (38) with the literature's previous result (36) reveals an important new component, namely the covariance terms in (38). Qualitatively, these covariance terms are always negative, and hence, help mitigate the FPP (34) in our model (see Proposition 6 below). Quantitatively, we need to verify whether their values can be negative enough to make $\sigma^{ds,\Delta r}$ into the negative-valued domain, to be consistent with FPP. The verification constitutes of several separate checks, which are unified in Proposition 6.

(1) Feller's admissibility condition: For the square-root processes (35) under consideration, $X^i(t)$ will be strictly positive, almost surely when the following conditions hold (recall that $H_0^i \equiv 0\ \forall i$)

$$2K_0^{i,P} > H_1^i > 0, \quad i \in \{h,f\}, \tag{39}$$

where the first inequality is Feller's condition, the second is a regularity to make sure that the square-root operations $\sigma^{X^i} = \sqrt{H_1^i X^i}$ do not generate complex-valued volatilities for all admissible $X^i(t)$.

(2) Affine interest rates: As summarized in Sec. 3.4, the polynomial–exponential-affine SDF of the forms (28) or (37) generally leads to quotient rfr

$$
\begin{aligned}
r^i(X^i) &= -\frac{1}{dt} E_t^P\left[\frac{dM^{i,P}(X^i,t)}{M^{i,P}(X^i,t)}\right] \\
&= C^i\left(\frac{H_1^i(1-C^i)}{2} - K_0^{i,P}\right)\frac{1}{X^i} + (\rho^i - K_0^{i,P}B^i - K_1^{i,P}C^i - B^iC^iH_1^i) \\
&\quad + B^i\left(-K_1^{i,P} - \frac{H_1^iB^i}{2}\right)X^i. \qquad (40)
\end{aligned}
$$

Hence, we impose the following specification constraint to enforce an affine rfr,[37]

$$K_0^{i,P} = \frac{H_1^i(1-C^i)}{2}, \quad i \in \{h,f\}. \qquad (41)$$

Substituting this specification into the interest rate in (40) gives

$$r^i = a^i + b^iX^i \quad \text{with} \quad \begin{cases} a^i = \rho^i - K_0^{i,P}B^i - K_1^{i,P}C^i - B^iC^iH_1^i, \\ b^i = -B^i\left(K_1^{i,P} + \dfrac{H_1^iB^i}{2}\right), \end{cases} \quad i \in \{h,f\}. \qquad (42)$$

(3) Non-negative interest rates: because $X^i(t)$ are strictly positive (possibly unbounded), like before, the conditions to assure the positivity for rfr $r^i = a^i + b^iX^i$ are $b^i > 0\ \forall i$ or

$$B^i\left(K_1^{i,P} + \frac{H_1^iB^i}{2}\right) < 0, \quad i \in \{h,f\}. \qquad (43)$$

We now combine these constraints to specify a construction that is consistent with the elusive negative sign of covariance (38) and the FPP (34). We recall that, as a prerequisite, the parametric setting is presumably symmetric (and independent) between countries, so similar parametric conditions are to be enforced in both countries.[38] A plausible sufficient condition for (38), and hence the FPP (34), is to make sure that the negative covariance terms $\mathrm{Cov}(X^i, \frac{1}{X^i})$ provide the dominant contribution to (38)

$$(C^i)^2H_1^i \gg b^i \gg (B^i)^2H_1^i, \quad i \in \{h,f\}.$$

[37] Note that the same specification gives rise to the affine rfr of Construction 1 of Sec. 3.1 (see the discussion following Eq. (28)).

[38] This symmetry and independence assumption follows from Backus *et al.* (2001) setting, and aims to place our construction on par with theirs to facilitate a comparison.

Combining this with conditions (39), (41), and (43), identifies a functional SDF construction that is consistent with the FPP.

Proposition 6. *Consider an international asset pricing model with the following features*

(i) *polynomial–exponential-affine functional SDF in physical measure,*[39]

$$M^{i,P}(X^i, t) \sim e^{-\rho t} e^{B^i X^i} (X^i)^{C^i}, \quad i \in \{h, f\}, \tag{44}$$

(ii) *affine, independent and symmetric (across countries) state dynamics* (35),

(iii) *additional parametric specifications*

$$\begin{aligned} &B^i > 0; \quad C^i < 0; \quad H_1^i > 0; \quad K_0^{i,P} = \frac{H_1^i(1 - C^i)}{2}; \\ &-K_1^{i,P} \gg B^i H_1^i, \quad i \in \{h, f\}. \end{aligned} \tag{45}$$

In this model, the change in exchange rates correlates negatively with interest rate differential (34), *i.e., the FPP holds.*

We elaborate on how these conditions are formulated in the first place, and refer to Appendix A for a technical proof of this proposition. First, the assumption of countries' independent risk factors breaks the FPP into separate intra-country anomalies in (38). That is, as long as (squared) mpr $(\eta^{i,QP})^2$ and interest rate r^i tend to negatively correlate for each country i, the assumed independence between countries' factors facilitates a simple summation of these negative correlations. These contributions sum up to enforce the deviation between changes in (squared) mpr differential $(\eta^{h,QP})^2 - (\eta^{f,QP})^2$ from those in the interest rate differential $r^h - r^f$ (34).

Second, the relation $H_1^i > 0$ is dictated by the Feller's admissibility condition (39) to assure the positivity for the volatility of the square-root process X^i. Equation (45) also implies $K_0^{i,P} > 0$, $K_1^{i,P} < 0$, i.e., state variable $X^i(t)$ is indeed mean-reverting in each country. Similarly, $K_0^{i,P} = \frac{H_1^i(1-C^i)}{2}$ is required to generate affine rfr r^i (40). The condition $C^i < 0$ is motivated by an economic intuition that $-C^i > 0$ characterizes the risk aversion of the representative agent in country $i \in \{h, f\}$ (when the SDF M^i is identified with the agent's marginal utility, see next). The condition $B^i > 0$, together with small absolute value $|B^i|$, assures non-negative interest rate (43). Therefore, within affine rfr class (41), (42), our functional SDF construction is both consistent

[39] This is (37), with $H_0^i \equiv 0$. Consequently, note that though parameters $\{H_1^i\}_{i=h,f}$ do not appear in SDFs $\{M^{i,P}\}_{i=h,f}$, they still contribute fundamentally to the forward premium via their role in the volatilities of state variables.

and robust with respect to the forward premium anomaly, in the sense that the FPP-consistent specifications (45) are satisfied for a wide range of parameters detailed in (45).

We now discuss economic merits of the FPP-consistent model of Proposition 6. An advantage of our functional SDF construction is its connection to the utilitarian framework and the associated economic interpretation. To visualize this connection in a simple illustration, we hereafter assume that state variable $X(t)$ represents the consumption level with explicit consumption (growth) dynamics (35) (recall that $H_0^i \equiv 0$),[40]

$$\frac{dX^i(t)}{X^i(t)} = \frac{\mu^{X^i,P}}{X^i(t)}dt + \frac{\sigma^{X^i}}{X^i(t)}dZ^i(t) = \left(\frac{K_0^{i,P}}{X^i(t)} + K_1^{i,P}\right)dt + \sqrt{\frac{H_1^i}{X^i(t)}}dZ^i(t). \tag{46}$$

In this simple illustration, $M^{i,P}(X,t)$ represents the agent's marginal utility. The consistence with FPP then relies fundamentally on the mean-reverting dynamics of the consumption $X(t)$, and the polynomial–exponential-affine specification of $M^P(X,t)$.

First, the associated utility function also has an appeal of exponential and power preferences, which is quantified by the implied risk aversion,[41]

$$\gamma^i(X^i,t) \equiv -\frac{X^i M_X^{i,P}(X^i,t)}{M^{i,P}(X^i,t)} = -C^i - B^i X^i. \tag{47}$$

The constant term is from the power component, and the linear term from the exponential component of the preference. Condition $B^i K_0^{i,P} \ll C^i K_1^{i,P}$ implies that preferences are almost power utility. The consumptions X^i mean-reverts about positive long term averages $\frac{K_0^{i,P}}{-K_1^{i,P}}$, which are well below $\frac{-C^i}{B^i}$, and risk aversions γ^i are positive.[42] Therefore, the marginal utilities $M^{i,P}$ decrease in X_i, and we conventionally associate a large value of X^i (i.e., low $M^{i,P}$) with good states of country i's economy and vice versa. As such, both consumption expected growth $\frac{\mu^{X^i,P}}{X^i}$ and growth volatility $\frac{\sigma^{X^i}}{X^i}$ drop in

[40] This identification is simplistic, but can easily be enriched by adding other (non-consumption) risk factors to the state space of $\{X_t\}$.

[41] Representative agent's marginal utility is (44), $U_X^i(X^i,t) = M^{i,P}(X^i,t) = e^{-\rho^i t} e^{B^i X^i} (X^i)^{C^i}$, $i \in \{h,f\}$, the associated (time additive) utility is $U^i(X^i,t) \sim e^{-\rho^i t} \int^{X^i} e^{-\rho^i t} e^{B^i Y} Y^{C^i} dY$.

[42] Utilities are concave, and representative agents have decreasing risk aversion γ^i ($B^i > 0$) for all admissible consumptions ($X^i > 0$) (47). This downward-sloping behavior of $\gamma^i(X^i)$ is due to exponential factor of preference.

good states. These dynamics are key to the FPP-consistency of the model underlying Proposition 6.

Second, for small enough B^i, which is relevant for the holding of condition (45), risk-averse representative agents demand lower risk premia in better states of economies. That is, market prices of risk $\eta^{QP,i}(X^i)$ decrease with consumption levels X^i,

$$\frac{\partial}{\partial X^i}\eta^{QP,i}(X^i) = \frac{\partial}{\partial X^i}\left(\gamma^i(X^i,t)\frac{\sigma^{X^i}(X^i)}{X^i}\right) < 0.$$

The inequality results from properties of our construction (45) that both risk aversion γ^i and consumption growth volatility $\frac{\sigma^{X^i}}{X^i} \sim \frac{1}{\sqrt{X^i}}$ drops, in good states.

Third, in light of consumption risk, the interest rate (40) has standard consumption-based components stemming from the intertemporal consumption smoothing (square-bracket expression) and precautionary savings (curly-bracket expression, $\theta^i \equiv -X^i \frac{M^{i,P}_{XX}}{M^{i,P}_X}$)

$$\begin{aligned} r^i(X^i) &= \rho^i + \left[\frac{\mu^{X^i,P}(X^i)}{X^i}\gamma^i(X^i)\right] - \left\{\frac{1}{2}\frac{[\sigma^{X^i}(X^i)]^2}{(X^i)^2}\gamma^i(X^i)\theta^i(X^i)\right\} \\ &= \text{constant} + \left[\frac{-C^iK^{i,P}_0}{X^i} - B^iK^{i,P}_1X^i\right] \\ &\quad - \frac{1}{2}H^i_1\left\{(B^i)^2X^i + \frac{C^i(C^i-1)}{X^i}\right\}. \end{aligned} \tag{48}$$

We observe that all C^i-terms originated from the power component, and all B^i-terms from the exponential component of the preference. There is an interesting interplay between intertemporal consumption smoothing desire (square-bracket terms), precautionary saving motives (curly-bracket terms), and the mean-reverting consumption dynamics, which gives rise to the FPP-consistency of equilibrium interest rates. Specifically, positive shocks in X^i make the representative agent more tolerant (elastic) of substituting consumption intertemporally (EIS $\frac{1}{\gamma^i(X^i)}$ increases). But as investors face negative expected consumption growth ($K^{i,P}_1 < 0$ in $\frac{\mu^{X^i,P}}{X^i}$ (46)), the boost in the elasticity of intertemporal substitution actually induces the agent to save less, which contributes to a surge in interest rates in good states (large X^i). Furthermore, when preference is mostly of power type ($B^i \ll C^i$), we may omit the second-order term of precautionary savings (term $\frac{1}{2}(B^i)^2H^i_1X^i$). Therefore, in our specification (45), the intertemporal consumption smoothing effect dominates over precautionary saving motives, and interest rate is procyclical (r^i increases with X^i).

Altogether, the last two points tell a consumption-based story on a negative correlation between rfr and risk premium $\mathrm{Cov}(r^i(t), \eta^{QP,i}(t)) < 0$ in (34), as fluctuations in consumption $X^i(t)$ move these two quantities in opposite directions. Looking back, Backus *et al.* (2001) consider a dynamic setting that can be implied from pure exponential preference (13). Our construction inherits this structure, but also enriches it with a dominant power factor (37), as a result of which the construction becomes consistent with the FPP. It might appear that power utility is all we need for the story to work here, and in particular it might also be tempting to set $B^i = 0$ to eliminate the exponential component of the preference. However, the important and relevant role of the exponential component $e^{B^i X^i}$ of SDF (44) is in generating the interest rate variability: slope coefficient b^i (42) is proportional to B^i. Due to the independence of countries' factors, there is no consumption risk sharing at an international level in the model of Proposition 6. In this aspect, the construction is a rather simplistic. Nevertheless, it provides a simple demonstration of the functional SDF approach, in constructing economic models consistent with pricing anomalies. Introducing interdependence between countries' risk factors will enrich the model, but might also obscure the simplicity of our illustration (Appendix B).

6. Conclusion

This paper starts with a simple observation that, when the SDF is a function of underlying state variables, the SDF satisfies a linear differential equation specified by risk-neutral state dynamics and the rfr. As a result, all possible and consistent SDFs can be determined from this differential equation. Then, the state dynamics in physical measure are followed, once a SDF solution has been determined. Accordingly, we propose a new, tractable and general functional SDF construction approach that is consistent with the given risk-neutral state dynamics. Functional SDF offers a framework to unify several existing asset pricing models. To illustrate, we establish simple conditions under which several popular settings of DTSMs of interest rates (affine, quadratic and quotient interest rate models), as well as pricing models based on LG processes, can be derived from functional SDFs.

As an application, we construct an international asset pricing model featuring functional SDFs that is consistent with the FPP and affine state dynamics. Intuitively, when the home consumption goes up and the home economy is in a good state, home bonds become cheaper and the home interest rate increases. At the same time, risk-averse international investors pay lower prices for home consumption risks and value the home currency more favorably. Altogether, these movements are consistent with the FPP, i.e., the home currency tends to appreciate when the home interest rate increases relative to the foreign interest rate.

Nomenclature

The following table lists all key quantities and their notations employed in the main text.

Notation	Description
P	physical (i.e., data-generating) measure
Q	risk-neutral measure
R	any general measure equivalent to P and Q
Z^P, Z^Q, Z^R	standard Brownian motions under respective measure
X	(vector) state variable
$\mu^{X,P}$, $\mu^{X,Q}$, $\mu^{X,R}$	dynamics (drift) of state variable X in measure P, Q, R, respectively
σ^X	dynamics (volatility) of state variable X (identical in any equivalent measure)
M^P, M^Q, M^R	stochastic discount factor (SDF) under respective measure
$M^P(X,t)$	SDF as proper function of (X,t)
$M^{P,\{\lambda_1,\lambda_2\}}(X,t)$	SDF as proper function of (X,t), parametrized by λ_1, λ_2
$\widehat{M}(v,t)$	Fourier transform of $M(X,t)$ (in variable X)
ρ	subjective discount factor
$\phi^P(X) = \frac{1}{M^P(X)} = \frac{e^{-\rho t}}{M^P(X,t)}$	reciprocal of SDF $M^P(X)$
$\xi^{QP} = \frac{dP^Q}{dP^P}$	Radon–Nikodym derivative to change measure from Q to P
$r(X,t)$	instantaneously risk-free rate (rfr) process
$\eta^{QP} = \lim_{dt\to 0} \frac{dZ^Q - dZ^P}{dt}$	market price of risk associated with measure change from Q to P
$\mathcal{D}^{X,P}$	infinitesimal operator associated with (diffusion) process X in measure P

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Appendix A. Proofs

We recall that subscripts always denote derivatives or partial derivatives (when appropriate); e.g., $f_X \equiv \frac{\partial f}{\partial X}$ throughout the paper. To simplify the notation, we also omit the explicit state and time contingency (X, t) from general function $f(X, t)$ wherever the omission does not create possible ambiguity.

Proof of Proposition 2. Our construction of functional SDF $M^P(t, X)$, out of given processes governing the state variables, is based on the key second-order linear differential equation (SOLDE) (8), and thus, benefits greatly from established mathematical results. A recent comprehensive resource on differential equations and their special function solutions is Olver *et al.* (2010).

In particular, Eq. (11) can be solved analytically by transforming it into the following standard confluent hypergeometric differential equation (CHGDE) (recall that subscripts denote the derivatives)

$$g_{zz} + \left(\frac{\gamma}{z} - 1\right) g_z - \frac{\delta}{z} g = 0, \tag{A.1}$$

whose two fundamental independent solutions are expressed in terms of confluent hypergeometric functions $\mathbf{\Phi}(.,.;z)$

$$\mathbf{\Phi}(\delta, \gamma; z); \quad z^{1-\gamma}\mathbf{\Phi}(\delta - \gamma + 1, 2 - \gamma; z).$$

That is, any solution $g^{\{\lambda_1,\lambda_2\}}(z)$ to (A.1) is a linear combination of the two independent solutions

$$g^{\{\lambda_1,\lambda_2\}}(z) = \lambda_1 \mathbf{\Phi}(\delta, \gamma; z) + \lambda_2 z^{1-\gamma}\mathbf{\Phi}(\delta - \gamma + 1, 2 - \gamma; z),$$

where λ_1, λ_2 are constants of integration. Specific steps to bring (11) into (A.1) are as follows.

First, after a change of variable

$$y = H_0 + H_1 X \leftrightarrow X = \frac{y}{H_1} - \frac{H_0}{H_1},$$

Equation (12) becomes

$$\phi^P_{yy} + \frac{2(K^Q_0 H_1 - K^Q_1 H_0 + K^Q_1 y)}{H_1^2 y}\phi^P_y + \frac{2\left(\rho + \frac{bH_0}{H_1} - a\right) - 2\frac{b}{H_1}y}{H_1^2 y}\phi^P = 0. \tag{A.2}$$

Next, we make the following transformation and another change of variable:

$$\phi(y) \equiv e^{\alpha y} g(\beta y); \quad z \equiv \beta y,$$

where α and β are two constants of choice to be determined below. Differential equation of $g(z)$ then follows from (A.2)

$$\beta^2 g_{zz} + 2\beta \left[\alpha + \frac{K_1^Q}{H_1^2} + \frac{\beta(K_0^Q H_1 - K_1^Q H_0)}{H_1^2 z} \right] g_z$$
$$+ \left[\left\{ \alpha^2 + \frac{2\alpha K_1^Q}{H_1^2} - \frac{2b}{H_1^3} \right\} + \frac{2\beta\left(\alpha[K_0^Q H_1 - K_1^Q H_0] + \rho + \frac{bH_0}{H_1} - a \right)}{H_1^2 z} \right] g = 0.$$

To bring this equation into the standard CHGDE (A.1), we choose parameter α such that the expression inside the curly brackets vanishes[43]

$$\alpha^2 + \frac{2\alpha K_1^Q}{H_1^2} - \frac{2b}{H_1^3} = 0 \Rightarrow \alpha = -\frac{K_1^Q}{H_1^2} \pm \left(\frac{(K_1^Q)^2}{H_1^4} + \frac{2b}{H_1^3} \right)^{\frac{1}{2}}. \quad \text{(A.3)}$$

Dividing both sides of above DE then yields

$$g_{zz} + \frac{2}{\beta} \left[\frac{2(K_0^Q H_1 - K_1^Q H_0)}{H_1^2 z} + \frac{2\left(\alpha + \frac{K_1^Q}{H_1^2} \right)}{\beta} \right] g_z$$
$$+ \left[\frac{2\left(\alpha[K_0^Q H_1 - K_1^Q H_0] + \rho + \frac{bH_0}{H_1} - a \right)}{\beta H_1^2 z} \right] g = 0.$$

Evidently, this equation is identical to standard CHGDE (A.1) by the following parameter identifications:

$$\gamma = \frac{2(K_0^Q H_1 - K_1^Q H_0)}{H_1^2}; \quad \beta = -2\left(\alpha + \frac{K_1^Q}{H_1^2} \right) = \mp 2 \left(\frac{(K_1^Q)^2}{H_1^4} + \frac{2b}{H_1^3} \right)^{\frac{1}{2}},$$

$$\delta = \frac{-2\left(\alpha[K_0^Q H_1 - K_1^Q H_0] + \rho + \frac{bH_0}{H_1} - a \right)}{\beta H_1^2}$$
$$= \frac{\alpha[K_0^Q H_1 - K_1^Q H_0] + \rho + \frac{bH_0}{H_1} - a}{\alpha H_1^2 + K_1^Q},$$

[43]Which root of α chosen should be dictated by economic considerations, such as how SDF $M^P(X)$ varies (increases or decreases) with state variable X. See Sec. 5 for an illustration.

where α is given by (A.3). Undoing previous transformations and changes of variables, we obtain the most general solution of (11)

$$\phi^{P,\{\lambda_1,\lambda_2\}}(X) = e^{\frac{\alpha}{\beta}z}[\lambda_1\Phi(\delta,\gamma;z) + \lambda_2 z^{1-\gamma}\Phi(\delta-\gamma+1, 2-\gamma; z)],$$

with $z = \beta y = \beta(H_0 + H_1 X)$. Finally, using definition (7), yields (12). □

Proof of Proposition 3. The specification of a completely affine DTSM with one factor $X \in R^+$ (Dai and Singleton, 2000) can be written as

$$\mu^{X,P} = K_0^P + K_1^P X; \quad (\sigma^X)^2 = X; \quad r = a + bX; \quad \eta^{QP} = \lambda_{11}\sqrt{X},$$

where λ_{11} is a constant. This specification implies that the dynamics are also affine under Q: $\mu^{X,Q} = \mu^{X,P} - \sigma^X\eta^{QP} = K_0^Q + K_1^Q X$ with

$$K_0^Q = K_0^P; \quad K_1^Q = K_1^P - \lambda_{11}. \tag{A.4}$$

In this setting, SDF $M^P(X,t)$ satisfies a special version of (4) and (5)

$$\frac{1}{2}XM_{XX}^P + [K_0^P + K_1^P X]M_X^P + [a + bX - \rho]M^P(X) = 0,$$
$$M_X^P + \lambda_{11}M^P = 0 \Rightarrow M_{XX}^P = \lambda_{11}^2 M^P.$$

Plugging the second equation into the first, and identifying terms of order X^0 (constants) and X^1 in both sides, respectively, yields

$$a = \rho + \lambda_{11}K_0^P = \rho + K_0^P(K_1^P - K_1^Q),$$
$$b = \lambda_{11}\left(K_1^P - \frac{\lambda_{11}}{2}\right) = \frac{1}{2}\left[(K_1^P)^2 - (K_1^Q)^2\right].$$

Using λ_{11} from (A.4) we obtain the first set of identities in Proposition 3. (The first identity above can be attributed to a choice of discount factor ρ, and was omitted in the proposition.)

The specification of a completely affine DTSM with one factor $X \in R^+$ (Cheridito *et al.*, 2007) can be written as

$$\mu^{X,P} = K_0^P + K_1^P X; \quad (\sigma^X)^2 = X; \quad r = a + bX; \quad \eta^{QP} = \frac{\lambda_{01}}{\sqrt{X}} + \lambda_{11}\sqrt{X},$$

where $\lambda_{01}, \lambda_{11}$ are constants. This specification implies that the dynamics are also affine under Q: $\mu^{X,Q} = \mu^{X,P} - \sigma^X\eta^{QP} = K_0^Q + K_1^Q X$ with

$$K_0^Q = K_0^P - \lambda_{01}; \quad K_1^Q = K_1^P - \lambda_{11}. \tag{A.5}$$

In this setting, SDF $M^P(X,t)$ satisfies a special version of (4) and (5)

$$\frac{1}{2}XM_{XX}^P + [K_0^P + K_1^P X]M_X^P + [a + bX - \rho]M^P(X) = 0,$$

$$M_X^P + \left(\frac{\lambda_{01}}{X} + \lambda_{11}\right) M^P = 0 \Rightarrow M_{XX}^P = \left(\frac{\lambda_{01} + \lambda_{01}^2}{X^2} + \frac{2\lambda_{01}\lambda_{11}}{X} + \lambda_{11}^2\right) M^P.$$

Plugging the second equation into the first, and identifying the terms of order X^{-1}, X^0 and X^1 in both sides, respectively, yields

$$0 = \lambda_{01}\left(K_0^P - \frac{\lambda_{01} + 1}{2}\right),$$
$$a = \rho + \lambda_{11} K_0^P + \lambda_{01} K_1^P - \lambda_{01}\lambda_{11},$$
$$b = \lambda_{11}\left(K_1^P - \frac{\lambda_{11}}{2}\right).$$

Finally, using λ_{01}, λ_{11} from (A.5), we obtain the second set of identities in Proposition 3. (The second identity above can be attributed to a choice of discount factor ρ, and was omitted in the proposition.)

Generally, the functional SDF in Construction 1 does not necessary imply P-affine dynamics. Conversely, a completely (or extended) affine DTSM does not necessarily imply a proper functional SDF $M^P(X,t)$. Only with these additional parameter restrictions, the functional SDF $M^P(X,t)$ in Construction 1 generates a completely (or extended) affine DTSM. □

Deriving rfr 18 in Construction 2. Note that interest rate is negative growth rate (drift) of SDF in any equivalent measure. Using Itô's lemma on function $M^R(X,t)$ determines this drift,

$$\begin{aligned}
&- M^R(X,t)\left[r(X)\,dt + \frac{\mu^{X,R}(X) - \mu^{X,Q}(X)}{\sigma^X(X)}\,dZ^R\right] \\
&\quad = dM^R(X,t) \\
&\quad = \left[\frac{1}{2}(\sigma^X(X))^2 M_{XX}^R(X,t) + \mu^{X,R}(X) M_X^R(X,t) - \rho M^R(X,t)\right] dt \\
&\qquad + \sigma^X M_X^R(X,t)\,dZ^R.
\end{aligned} \tag{A.6}$$

The substitution of specified R-dynamics in Construction 2 then yields the interest rate (18).

Proof of Proposition 4. Equation (20) can be solved analytically by transforming it into a form of standard Weber differential equation (WDE)

$$g_{zz} - \left(\frac{z^2}{4} + \nu\right) g = 0, \tag{A.7}$$

whose two fundamental independent solutions are expressed in terms of confluent hypergeometric functions $\mathbf{\Phi}(.,.;z)$

$$e^{\frac{-z^2}{4}}\mathbf{\Phi}\left(\frac{\nu}{2}+\frac{1}{4},\frac{1}{2};\frac{z^2}{2}\right); \quad ze^{\frac{-z^2}{4}}\mathbf{\Phi}\left(\frac{\nu}{2}+\frac{3}{4},\frac{3}{2};\frac{z^2}{2}\right).$$

That is, any solution $g^{\{\lambda_1,\lambda_2\}}(z)$ to (A.7) is a linear combination of the two independent solutions

$$g^{\{\lambda_1,\lambda_2\}}(z) = \lambda_1 e^{\frac{-z^2}{4}}\mathbf{\Phi}\left(\frac{\nu}{2}+\frac{1}{4},\frac{1}{2};\frac{z^2}{2}\right) + \lambda_2 ze^{\frac{-z^2}{4}}\mathbf{\Phi}\left(\frac{\nu}{2}+\frac{3}{4},\frac{3}{2};\frac{z^2}{2}\right),$$

where λ_1, λ_2 are constants of integration. Specific steps to bring (20) into (A.7) are as follows.

First, after a transformation

$$\phi^P(X) \equiv e^{MX+NX^2}g(X),$$

where M, N are constants of choice to be determined below, Eq. (20) becomes

$$\frac{1}{2}H_0 g_{XX} + \left[(2H_0N + K_1^Q)X + (H_0M + K_0^Q)\right]g_X - \left[AX^2 + BX + C\right]g = 0, \tag{A.8}$$

where parameters A, B, C are related to M, N and are deferred till after the latter are determined. Evidently, to bring (A.8) into Weber form (A.7), we choose M, N such that term g_X vanishes

$$H_0M + K_0^Q = 0 \Rightarrow M = -\frac{K_0^Q}{H_0},$$
$$2H_0N + K_1^Q = 0 \Rightarrow N = -\frac{K_1^Q}{2H_0}.$$

These choices then pin down A, B, C in Eq. (A.8)

$$A = -\frac{4H_0N^2 + 4K_1^QN - 2\rho_2}{H_0} = \frac{(K_1^Q)^2 + 2\rho_2H_0}{H_0^2},$$

$$B = -\frac{4H_0N + 4K_0^QN + 2K_1^QM - 2\rho_1}{H_0} = \frac{2(K_1^QH_0 + 2K_0^QK_1^Q + \rho_1H_0)}{H_0^2},$$

$$C = -\frac{H_0M^2 + 2K_0^QM + 2(\rho - \rho_0)}{H_0} = \frac{(K_0^Q)^2 - 2H_0(\rho - \rho_0)}{H_0^2}.$$

Next, the change of variable[44]

$$z = (4A)^{\frac{1}{4}}\left(X + \frac{B}{2A}\right),$$

transforms (A.8) into

$$g_{zz} - \left(\frac{z^2}{4} + \frac{4AC - B^2}{4A^{\frac{3}{2}}}\right)g = 0.$$

Identifying this with standard Weber equation (A.7)

$$\nu \equiv \frac{4AC - B^2}{4A^{\frac{3}{2}}},$$

immediately yields the most general solution of the original equation (20)

$$\phi^P(X) = e^{MX+NX^2}\left[\lambda_1 e^{\frac{-z^2}{4}}\mathbf{\Phi}\left(\frac{\nu}{2} + \frac{1}{4}, \frac{1}{2}; \frac{z^2}{2}\right) + \lambda_2 z e^{\frac{-z^2}{4}}\mathbf{\Phi}\left(\frac{\nu}{2} + \frac{3}{4}, \frac{3}{2}; \frac{z^2}{2}\right)\right],$$

where λ_1, λ_2 are constants of integration, and $z = (4A)^{\frac{1}{4}}(X + \frac{B}{2A})$ is linear in the original state variable X. Finally, using definition (7) yields (21). □

Proof of Proposition 5. This proof is similar to that of Proposition 2 in the sense that Eq. (24) can also be transformed into the standard CHGDE (A.1), though detailed steps are a bit different.

First, after a transformation

$$\phi^P(X) \equiv e^{\alpha X} X^\beta g(X)$$

where α, β are constants of choice to be determined below, Eq. (24) becomes

$$g_{XX} + \left[\frac{2}{X}\left(\beta + \frac{K_0^Q}{H_1}\right) + 2\left(\alpha + \frac{K_1^Q}{H_1}\right)\right]g_X + \left[\frac{\beta^2 H_1 + (2K_0^Q - H_1)\beta - 2\theta_{-1}}{H_1 X^2} + \frac{2(H_1\alpha\beta + K_0^Q\alpha + K_1^Q\beta + \rho - \theta_0)}{H_1 X} + \frac{H_1\alpha^2 + 2K_1^Q\alpha - 2\theta_1}{H_1}\right]g = 0.$$

To bring this into (A.1), we choose parameters α, β such that coefficients of the order X^0 and X^{-2} of term g (last term in the above differential equation) vanish,

$$H_1\alpha^2 + 2K_1^Q\alpha - 2\theta_1 = 0 \Rightarrow \alpha = \frac{-K_1^Q \pm [(K_1^Q)^2 + 2H_1\theta_1]^{1/2}}{H_1},$$

[44] Real value for z requires $A > 0$. In case of $A < 0$, we can proceed similarly to bring the original equation (20) to another form of WDE $g_{zz} + (\frac{z^2}{4} - \nu)g = 0$.

$$\beta^2 H_1 + (2K_0^Q - H_1)\beta - 2\theta_{-1} = 0$$
$$\Rightarrow \beta = \frac{H_1 - 2K_0^Q \pm [(2K_0^Q - H_1)^2 + 8H_1\theta_{-1}]^{1/2}}{2H_1}.$$

The differential equation for g becomes

$$g_{XX} + \left[\frac{2}{X}\left(\beta + \frac{K_0^Q}{H_1}\right) + 2\left(\alpha + \frac{K_1^Q}{H_1}\right)\right] g_X + \left[\frac{2(H_1\alpha\beta + K_0^Q\alpha + K_1^Q\beta + \rho - \theta_0)}{H_1 X}\right] g = 0.$$

Finally, the change of variable

$$z = -2\left(\alpha + \frac{K_1^Q}{H_1}\right) X = \mp \frac{2\left[(K_1^Q)^2 + 2H_1\theta_1\right]^{1/2}}{H_1} X,$$

precisely transforms the above equation for g into the standard CHGDE (A.1). Analytical solutions for $g(z)$ and then $\phi^P(X)$ follow similarly, as in the proof of Proposition 2. We thus obtain (25). □

Measure-invariance of LG models: We first note that, if M^R is the SDF in an equivalent measure, corresponding Radon–Nikodym derivatives $\xi^{RP} \equiv \frac{M^P(X,t)}{M^R(X,t)}$ and $\xi^{PR} \equiv \frac{M^R(X,t)}{M^P(X,t)}$ are P- and R-martingale, respectively. They have null drifts in respective measures, $\mathcal{D}^{X,P}\xi^{RP} = 0$, $\mathcal{D}^{X,R}\xi^{PR} = 0$. The last equation is (26) in Construction 3. To demonstrate the invariance of the LG pricing framework under a change of measure, we assume the LG structure (29) in an equivalent measure R

$$\mathcal{D}^{X,R}X(t) = -\Omega X(t); \quad M^R(X,t) = \lambda^m \cdot X(t); \quad M^R D(X,t) = \lambda^{md} \cdot X(t).$$

We define a new state variable, $\widehat{X}(t) \equiv \xi^{RP} X(t)$, where $\xi^{RP} = \frac{M^P}{M^R}$ is the Radon–Nikodym derivative characterizing the change between physical measure P and equivalent measure Q (ξ^{RP} is a scalar P-martingale). We can bring the above specifications to physical measure P

$$\begin{aligned}
\mathcal{D}^{X,P}\widehat{X}(t) &= \mathcal{D}^{X,P}[\xi^{RP}X(t)] = \xi^{RP}\mathcal{D}^{X,R}[X(t)] = -\xi^{RP}\Omega X(t) = -\Omega\widehat{X}(t),\\
M^P(X,t) &= \frac{M^P(X,t)}{M^R(X,t)} M^R(X,t) = \xi^{RP}\lambda^m \cdot X(t) = \lambda^m \cdot \widehat{X}(t),\\
M^P D(X,t) &= \frac{M^P(X,t)}{M^R(X,t)} M^R D(X,t) = \xi^{RP}\lambda^{md} \cdot X(t) = \lambda^{md} \cdot \widehat{X}(t).
\end{aligned} \tag{A.9}$$

Clearly, $X(t)$ is LG process under P if and only if $\widehat{X}(t)$ is LG process under R. This measure-invariant property shows that LG pricing dynamics are already most general, and thus, neutral, with respect to measure rotation.

Proof of Proposition 6. We will show that when relations specified in (45) hold, the covariance $\sigma^{ds,\Delta r} < 0$, or high interest rate currencies tend to appreciate, i.e., FPP prevails. Using delta method approximation, we rewrite the key FPP-consistent condition (38) as[45]

$$\sigma^{ds,\Delta r} \approx b^h \mathrm{Var}(X^h)\left[b^h + \frac{1}{2}(B^h)^2 H_1^h - \frac{(C^h)^2}{2}\frac{(K_1^{h,P})^2}{(K_0^{h,P})^2}H_1^h\right] + (h \leftrightarrow f) < 0.$$

Since $b^h, b^f > 0$ (see (42) and (43)), this covariance is negative when the last term inside square brackets dominates the first two terms (we explicitly plug in $K_0^{i,P} = \frac{H_1^i(1-C^i)}{2}$ and $b^i = -B^i(K_1^{i,P} + \frac{B^i H_1^i}{2})$ (42) in what follows)

$$\frac{(C^i)^2(K_1^{i,P})^2}{H_1^i(1-C^i)^2} \gg -B^i\left(K_1^{i,P} + \frac{B^i H_1^i}{2}\right) \gg (B^i)^2 H_1^i \quad i \in \{h, f\}.$$

Since $B^i, H_1^i > 0$, the last relation of (45) $-K_1^{i,P} \gg B^i H_1^i$ clearly implies the above inequalities, and thus, also the negative covariance $\sigma^{ds,\Delta r} < 0$, for any risk aversion coefficient $-C^i$, that is, strictly positive. The latter part of Sec. 5.2 justifies all relations in (45), we have just derived here from a structural (risk-based) consideration. □

Appendix B. Multi-factor Settings

With the exception of Sec. 4.2, our SDF constructions have been demonstrated in one-factor settings wherein state variable $X(t)$ is a one-dimensional (scalar) process. In this section, we briefly show that the functional SDF approach can also be extended to multiple-factor settings, which gives modeling flexibility to the approach.

We consider a vector-valued diffusion state variable $X(t)$ in $\mathbf{R}^n$, driven by m, independent of standard Brownian motions $Z(t) \in \mathbf{R}^m$. To illustrate the main ingredients of the multi-factor settings, below we work with the basic construction (Sec. 2.2, generalizations to other constructions follow in analogy). In this premise, the state dynamics specification in risk-neutral measure (n-vector drift $\mu^{X,Q}(X)$, $n \times m$-matrix volatility $\sigma^X(X)$) and scalar rfr

[45] The delta method approximation gives $\mathrm{Cov}(X^i(t), \frac{1}{X^i(t)}) \approx -\frac{\mathrm{Var}(X^i(t))}{(E[X^i(t)])^2}$, $i \in \{h, f\}$. We also replace $E^P[X^i]$ by its long-run mean $\frac{-K_0^{i,P}}{K_1^{i,P}}$ in this approximation.

process $r(X)$ are given, while maintaining Assumption 1 on scalar-valued functional SDF $M^P(X,t) = e^{-\rho t}M^P(X)$. We apply Itô's lemma on function $M^P(X,t)$ and identify it with the martingale differential representation (2), which results in a second-order partial differential equation (PDE) — a multi-factor counterpart of (6)

$$\frac{1}{2}\operatorname{Tr}(\sigma^X(X,t)\sigma^{X,T}(X,t)M^P_{XX}(X)) + \mu^{X,Q}(X)\cdot M^P_X(X) + \left(r(X) - \frac{\operatorname{Tr}\left[M^{P,T}_X(X)\sigma^X(X)\sigma^{X,T}(X)M^P_X(X)\right]}{(M^P(X))^2} - \rho\right)M^P(X) = 0,$$

where n-vector M^P_X and $n \times n$-matrix M^P_{XX} denote the gradient and Hessian of M^P. The change of variable $\phi^P(X) \equiv \frac{1}{M^P(X)}$ transforms the above non-linear PDE into a linear one (similar to (8))

$$\frac{1}{2}\operatorname{Tr}(\sigma^X(X,t)\sigma^{X,T}(X,t)\phi^P_{XX}(X)) + \mu^{X,Q}(X)\cdot\phi^P_X(X) + (\rho - r(X))\phi^P(X) = 0. \tag{B.1}$$

We note a practical advantage of the multi-factor construction. First, the gist of this construction (Proposition 1) is that any positive solution of differential equation (B.1) can be a (inverse of) SDF, consistent with the given state dynamics. Therefore, we do not need to solve this PDE in full generality. Identifying special solutions of desired properties of (B.1) suffices for many practical modeling purposes. One also has more flexibilities to impose structural constraints on the selected solutions in multi-factor settings. We illustrate this modeling advantage in a simple specific example, via a separation of variables.

We consider a setting with two correlating factors $(X,Y)^T$. Similar to Construction 1 in Sec. 3.1, we specify an affine interest rate and Q-affine state dynamics

$$r = a + b^x X + b^y Y; \quad \begin{pmatrix} dX \\ dY \end{pmatrix} = \mu^Q dt + \sigma\begin{pmatrix} dZ^{X,Q}(t) \\ dZ^{Y,Q}(t)\end{pmatrix}, \tag{B.2}$$

where $Z^{X,Q}(t)$, $Z^{Y,Q}(t)$ are uncorrelated standard Brownian motions in risk-neutral measure Q and

$$\mu^Q \equiv \begin{pmatrix} \mu^{X,Q} \\ \mu^{Y,Q} \end{pmatrix} = \begin{pmatrix} k^x_0 + k^{xx}_1 X \\ k^y_0 + k^{yx}_1 X + k^{yy}_1 Y \end{pmatrix},$$

$$\sigma\sigma^T \equiv \begin{pmatrix} \sigma^{2xx} & \sigma^{2xy} \\ \sigma^{2yx} & \sigma^{2yy} \end{pmatrix} = \begin{pmatrix} h_0^{xx} + h_1^{xx} X & h_0 + h_1 X \\ h_0 + h_1 X & h_0^{yy} + h_1^{yx} X + h_1^{yy} Y \end{pmatrix}.$$

Substituting these specifications into Eq. (B.1), we find special solutions of interest via a separation of variables. Specifically, for solutions of the form $\phi(X, Y) \sim e^{BY} G(X)$. This is indeed a solution of (B.1) if the following relations hold for unknowns B and $G(X)$:

$$h_1^{yy} B^2 + k_1^{yy} B - b^y = 0,$$

$$\frac{1}{2}\sigma^{2xx} G_{XX} + (B\sigma^{2xy} + \mu^{X,Q}) G_X + \left(\frac{B^2}{2}[h_0^{yy} + h_1^{yx} X] + B[k_0^y + k_1^{yx} X] + \rho - a - b^x X\right) G = 0.$$

Solving first the quadratic equation yields parameter B. With all coefficients being affine in X, the second equation is identical to (11) of Construction 1. The most general solution of $G(X)$ is in term of the confluent hypergeometric functions $\Phi(.,.;\beta X)$, as readily given in Proposition 2. Accordingly, in practice, we may start out with a functional SDF of the class

$$M^P(X, Y, t) = \frac{e^{-\rho t} e^{-BY - \alpha X}}{\lambda_1 \Phi(\delta, \gamma; \beta X) + \lambda_2 (\beta X)^{1-\gamma} \Phi(\delta - \gamma + 1, 2 - \gamma; \beta X)},$$

which will be consistent with multi-factor dynamics (B.2). Yet different specific choices within this class yield a rich set of possible equilibrium interpretations, market prices of risk, and P-dynamics as we have seen in our re-constructions of various term structure models, linearity-generating dynamics, and specially, the consistency of the FPP.

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