Entangled Risks in Incomplete FX Markets*

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Abstract

We reconcile the international correlation of macroeconomic consumptions, the Backus-Smith puzzle, and currency excess returns by uncovering and employing a novel market-based mechanism of risk entanglement, i.e., the specific configuration in which risks are embedded in incomplete international asset markets. When risks are entangled in FX markets, there exist multiple pricing-consistent exchange rates, but none of them is necessarily equal to the ratio of the given stochastic discount factors (SDFs) or their projectors. Therefore, risk entanglement decouples the exchange rate dynamics from those of cross-country relative pricing. Pursuing this decoupling, we calibrate a simple risk entanglement setup to identify market settings that consistently accommodate these three regularities of international finance.

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1 Introduction

The current paper introduces a novel market-based notion of risk entanglement, and employs it to address three regularities of international financial markets, namely the mild international correlation of macroeconomic quantities, sizable premia on currency strategies, and the low implied correlation between the exchange rate and the ratio of stochastic discount factors (SDFs). Risk entanglement arises when asset return risks are not individually contracted in perfectly integrated but incomplete international markets so that observing asset prices in all denomination currencies is insufficient to unambiguously pin down prices of risks in any currency. Risk entanglement characterizes the intertwined embedding of multiple risks into few traded assets, and substantially weakens the implications of international asset prices on the pricing of risks in different currencies. Conceptually, risk entanglement lowers the large SDF correlation that is needed to account jointly for smooth exchange rates and high equity premia in complete market settings, yielding a new market-based perspective to rationalize international asset pricing puzzles.

Understanding the rationales behind these regularities is important because they are some of the most fundamental implications of the basic benchmark model of international asset pricing. The benchmark model posits a strong relationship between the exchange rate and cross-country relative pricing dynamics, with the latter being quantified by either the ratio of stochastic discount factors (SDFs) in complete markets, or the ratio of SDF projectors in incomplete markets. In particular, given smooth exchange rates and volatile SDFs, the model posits high correlations between SDFs, between the exchange rate and the SDF ratio, and takes no stance on the profitability of currency strategies based on interest rates. Yet, under standard time separable preference specifications, the data paint a starkly different picture. The international correlation puzzle by Brandt et al. (2006) documents mild correlations between cross-country consumptions, and hence, between the implied SDFs. The Backus and Smith (1993) puzzle documents insignificant correlations between the exchange rate and consumption ratio, and hence, between the exchange rate and the implied SDF ratio. The currency literature documents profitable strategies of currencies sorted on interest rates.

Risk entanglement sheds new light on these prominent puzzles by decoupling exchange rate dynamics from that of the cross-country relative pricing (i.e., ratio of SDFs and SDF projectors). We define risks as entangled if there exist risks that affect asset returns but cannot be singly replicated by a portfolio of traded assets. Intuitively, risk entanglement characterizes the specificity in which risks are embedded in traded assets. Given country-specific SDFs and the associated pricing dynamics, this specificity in the risk-asset configuration matters crucially for the determination and possible multiplicity of pricing-consistent exchange rates. This is because when risks are not singly traded but entangled in asset returns, they can be construed and priced differently in the foreign currency by different possible exchange rate choices, while keeping all traded assets (as claims contingent on compounded risks) consistently priced by the given SDFs.[1] In sharp contrast, when risks are completely disentangled, every risk in conditional asset returns is singly traded in markets, while markets can still be incomplete due to unhedged risks in SDFs or in conditional moments of asset returns. Under this premise, along every risk dimension of the conditional asset return space, the exchange rate is unambiguously equal to the ratio of SDFs projected in that risk dimension. As a result, when risks are completely disentangled, the exchange rate is unique and reflects the relative pricing dynamics across the two currencies.

Our intuitive discussion above points to three novel insights of the risk entanglement framework, namely, (i) the deviation between the exchange rate and ratio of SDF projectors, (ii) multiple consistent exchange rates given a pair of country-specific SDFs, and (iii) every consistent exchange rate then identifies a distinct corresponding market configuration that gives rise uniquely to this exchange rate ex-post. Our modeling innovation is to introduce a portfolio representation of the exchange rate that fully quantifies the constraints from entangled risk pricing in FX markets on the exchange rate. We elaborate on the three insights in turn.

First, the deviation between the exchange rate and cross-country pricing dynamics speaks directly to a subtle but crucial enrichment of the benchmark model of international asset pricing. To the extent that the puzzles are construed on a strong benchmark relationship between the exchange rate and ratio of SDFs (or their projectors), by deviating from such a relationship, risk entanglement alleviates the principal roadblock of the previous literature to address these puzzles. Our paper is the first to identify risk entanglement as the source to this deviation in arbitrage-free, perfectly integrated and frictionless international financial markets. Incidentally, popular settings of complete markets, or incomplete markets with pure-diffusion risks, are special settings of completely disentangled risks, and hence overlook the solution approach based on risk entanglement.

Second, the multiplicity of pricing-consistent exchange rates reveals an ex-ante ambiguity (hence, flexibility) between prices and exchange rate when available traded assets are scarce. Intuitively,

[1] We prove an unambiguous but technical result that the exchange rate differs from the ratio of SDF projectors if and only if exchange rate risks are entangled in FX markets in Maurer and Tran (2018).

risk entanglement (i.e., many risks embedded in relatively few traded assets) weakens constraints from asset prices on the pricing of risks in different currencies, rendering multiple possible exchange rates. To model this market nature of risk entanglement into exchange rate dynamics, our setup takes as given the country-specific SDFs and a set of basis asset returns (specified in the home currency). When the foreign bond is a traded asset, its return in the home currency is proportional to the exchange rate and also is spanned by the basis returns. These observations provide a portfolio representation of the exchange rate growth in terms of basis asset returns. Employing this exchange rate, the pricing of basis assets in the foreign currency denomination then generate a non-linear equation system to solve for the portfolio weights in the exchange rate representation. Quantitatively, a higher degree of risk entanglement increases the non-linearity of the equation system, yielding a larger number of consistent exchange rate solutions.

Third, this multiplicity of pricing-consistent exchange rates presents a modeling advantage of the risk entanglement approach to rationalize exchange rate behaviors. Ex-post, obtaining multiple pricing-consistent exchange rates identifies a richer set of possible model-implied exchange rate dynamics to match its moments in the data. When a desirable exchange rate (among all solutions) is found based on this empirical ground, the associated portfolio representation of this exchange rate solution informs us the corresponding and specific asset market configuration. Ex-ante, if we start with this specific asset market configuration, then the only consistent exchange rate is the desirable one under consideration.

In sum, because risk entanglement weakens the relationship between exchange rate and country-specific relative pricing, it mitigates strong implications from the exchange rate volatility on the correlation between SDF projectors, and hence, the international correlation puzzle. Risk entanglement also speaks to the Backus-Smith puzzle by moderating the correlation between the exchange rate and the SDF ratio, and by implication, between the exchange rate and the consumption ratio. By incorporating the well documented currency excess returns as additional constraints in our setup, sufficient risk entanglement is able to accommodate these three regularities. For illustration, we calibrate the international correlation, the Backus-Smith puzzle, and currency excess returns in a single model featuring risk entanglement in FX markets.

Related Literature: Recent work by Lustig and Verdelhan (2018) and Bakshi et al. (2018) investigate the possible role of incomplete market settings in mitigating the international correlation puzzle by Brandt et al. (2006). These papers start with highly correlated SDFs, which price financial assets and imply a reasonable exchange rate volatility. They, then, add unspanned noise to the SDFs to reduce the correlation between them. Performing explicitly a qualitative analysis in log-normal risk settings, Lustig and Verdelhan (2018) conclude that the noise needed to resolve the correlation puzzle is unreasonably large in these settings and is at odds with empirical currency risk premia and the Backus-Smith puzzle. For non-log-normal distributions, they also present numerical evidence indicating difficulty to simultaneously calibrate the international correlation, currency risk premium, and the Backus-Smith puzzles. Taking a different modeling approach, Bakshi et al. (2018) consider international market segmentation. They discipline the amount of this unspanned noise by limiting the reward-to-risk ratio which a hypothetical asset written on this noise could earn to address the puzzle. Additionally, Bakshi and Crosby (2016) present arguments that a state-by-state relationship between exchange rate and country-specific pricing dynamics does not necessarily encapsulate all incomplete market possibilities.

Our setup does not assume market segmentation, state-by-state configuration between exchange rate and countries' SDFs, or log-normal distributions. In contrast, we employ risk entanglement as the key mechanism to conceptually and systematically weaken the link between the exchange rate and the cross-country relative pricing. In this respect, risk entanglement can also be seen as a dedicated algorithmic framework in which inputs are desired empirical constraints (e.g., large currency premia), and outputs are the needed risk-asset configurations. The risk entanglement framework succinctly captures contributions from all moments of the risk distribution, and reverse-engineers market configurations which the general but undedicated entropy search approach employed by Lustig and Verdelhan (2018) might have left out.

This paper is also related to the vibrant literature decoupling exchange rate dynamics from equilibrium consumptions through enriched structural features of SDFs. Those papers featuring incomplete or segmented international financial markets include Zapatero (1995), Sarkissian (2003), Dumas et al. (2003), Chaieb and Errunza (2007), Pavlova and Rigobon (2007) and Favilukis et al. (2015). Other papers employing rich preference features (e.g., recursive utilities and habit formations) while maintaining complete markets, include Colacito and Croce (2011), Bansal and Shaliastovich (2012), Colacito et al. (2018) and Stathopoulos (2017). Our approach instead employs asset risk entanglement to dilute the link between SDFs and the exchange rate. We take SDFs as given, hence abstracting from structural and equilibrium features giving rise to these SDFs. In one direction, the choice of these input SDFs needs to be guided by desirable structural features established in the literature. In the other direction, our analysis on the consistency between the given SDFs and data-matching exchange rate also informs structural models about desirable risk-asset configurations (i.e., risk entanglement) and SDFs. By extending risk-asset configurations to a novel class of risk entanglement, we effectively relieve the strain on the choice of qualified structural SDFs, yielding a larger flexibility to structural modeling consideration. Structural settings incorporating risk entanglements offer new perspectives in jointly modeling financial risks and exchange rates, and are subject of our future research.

On a technical note, Burnside and Graveline (2012) are the first to investigate the possibility that the exchange rate does not necessarily equal to the ratio of SDF projectors in incomplete market settings. Our exchange rate portfolio representation approach helps to concretely identify risk entanglement as the source of the deviation (between the exchange rate and ratio of SDF projectors) raised in their paper, and employ it to address and calibrate international finance puzzles.

Our paper is organized as follows. Section 2 illustrates the concept of risk entanglement in several simple market settings, discusses key underlying intuitions, and also states risk entanglement results in general settings. Section 3 models risk entanglement to calibrate the international correlation, currency premium, and the Backus-Smith puzzles. Section 4 concludes. Appendix A presents details on risk entanglement in general jump diffusion settings, Appendix B provides technical derivations and proofs, Appendix C elaborates on risk entanglement in discrete settings, Appendix D illustrates the construction of SDF projectors for diffusion risk settings.

2 Risk Entanglement: Illustrations and Formal Results

The concept and implications of risk entanglement in asset markets are most easily illustrated in specific examples. In this section, we first present simple market settings to introduce the basic procedure of the exchange rate determination, define risk entanglement, and demonstrate its effects through examples. We present general results of risk entanglement at the end of this section, and relegate derivations to Appendix A.

2.1 Market Setup and Risk Entanglement Definition

Throughout, we consider two countries (home and foreign) $I \in \{H, F\}$ and assume arbitrage-free, frictionless and fully integrated international financial markets. We take as given (i) countries' stochastic discount factors M_H, M_F, and (ii) returns specified in the home currency denomination of all traded assets. In particular, we assume that both home and foreign bonds are traded assets in international financial markets. We then illustrate the determination of all possible pricing-consistent exchange rates e by (i) converting all traded asset returns to the foreign currency denomination using e, and (ii) requiring that all asset returns denominated in the foreign currency be priced correctly by the foreign SDF M_F.

Specifically, consider a continuous time setting in which the risk space consists of one diffusion risk and one jump risk. The diffusion risk is characterized by a standard Brownian process Z_t, and the jump risk by a Poisson counting process $\mathcal{N}_t$ of arrival intensity λ_t. Hence, the jump takes place (i.e., Poisson counter increases by one $d\mathcal{N}_t \equiv \mathcal{N}_{t+dt} - \mathcal{N}_t = 1$) in the infinitesimal time interval $(t, t+dt)$ with probability λdt. Country I's SDF is given by,

$$\frac{dM_{It}}{M_{It}} \equiv \frac{M_{It+dt}}{M_{It}} - 1 = -r_I dt - \eta_I dZ_t + \left(e^{\Delta_I} - 1\right)\left(d\mathcal{N}_t - \lambda dt\right), \qquad I \in \{H, F\}, \tag{1}$$

where r_I, η_I and $(e^{\Delta_I} - 1)$ respectively are the risk-free rate, the price of diffusion risk and the price of jump risk in currency $I \in \{H, F\}$.[2]

Let $\{Y\}$ denote the set of traded assets. Without loss of generality we adopt the asset return specification from home investors' perspective. We take as given the return basis (specified in the home currency denomination) $\left\{\frac{dB_{Ht}}{B_{Ht}}, \frac{dY_{nt}}{Y_{nt}}\right\}$ that spans the space of all traded assets $\{Y\}$ in international financial markets, where B_H denotes the home bond, and Y_n the n-th risky asset. For notational convenience, we simply express the set of traded assets by its basis, $\{Y\} = \{B_H, Y_n\}$. Our exchange rate convention is that e_t denotes the amount of foreign currency that buys one unit of the home currency at time t. Due to the market integration assumption, foreign investors trade the same set of assets, but asset returns to them are those specified in the home currency multiplied by the exchange rate growth $\frac{e_{t+dt}}{e_t}$. Note that a traded basis asset in the set $\{Y\}$ may originate from either the home or foreign economy.[3] We look for the net exchange rate growth in the form of a general jump diffusion process,

$$\frac{de_t}{e_t} = \mu_e dt + \sigma_e dZ_t + \left(e^{\Delta_e} - 1\right)\left(d\mathcal{N}_t - \lambda dt\right), \tag{2}$$

[2] When a jump takes place, the SDF changes instantly from M_{It} to $M_{It}e^{\Delta_I}$. Therefore, Δ_I is the jump size of the SDF M_I. Note that quantities r_I, η_I, and Δ_I can be time-varying in our setting in general.

[3] Without loss of generality, the market integration allows us to specify its return conventionally in the home currency denomination.

buy solving for the mean μ_e, volatility σ_e, and jump size Δ_e of the exchange rate growth.

By construction, the home SDF M_H prices asset returns specified in the home currency denomination, while M_F prices returns on these same assets but in the foreign currency denomination. By combining the Euler pricing equations across the two currency denominations, we obtain a simple equation for every traded asset Y_i,[4]

$$Cov_t\left(\left[\frac{M_{Ht+dt}}{M_{Ht}}-\frac{M_{Ft+dt}}{M_{Ft}}\frac{e_{t+dt}}{e_t}\right]\ ,\ \frac{Y_{nt+dt}}{Y_{nt}}\right)=0, \qquad \forall Y_n. \tag{3}$$

The cross-currency pricing equations above are the key to determine all possible pricing-consistent exchange rates. These exchange rate solutions depend critically on how risks are embedded into the given traded asset returns $\left\{\frac{Y_{nt+dt}}{Y_{nt}}\right\}$. Risk entanglement characterizes the risk-asset embedding in asset markets, both qualitatively and quantitatively. We first present a formal definition of risk entanglement, then motivate its rationales in specific examples of this section, before deriving its properties in the next section.

Definition 1 (Risk Entanglement) *Risks are entangled in asset markets if there exist risks of significant magnitudes impacting instantaneous returns of traded assets* $\left\{\frac{dY_{t+dt}}{Y_t}\right\}$, $\forall Y\in\{Y\}$, *that cannot be singly replicated by the return on any portfolio of traded assets.*[5] *Otherwise, risks are completely disentangled in asset markets if every risk impacting instantaneous returns of traded assets is perfectly and singly replicated by the return on a portfolio of traded assets.*

In continuous time, risks of "significant magnitudes" refer to jump risks of significant jump sizes. Risk entanglement requires such risks because other risks (i.e., diffusion risks or jump risks of small jump sizes) are always completely disentangled in any asset markets.[6] We now illustrate this property and other aspects of this formal definition in three specific market configurations in turn: complete market, incomplete market without risk entanglement, and incomplete market with risk entanglement. The illustration introduces a portfolio representation approach to determine the

[4]These equations arise from subtracting Euler equations in the foreign currency from those in the home currency for each basis asset Y_n, $E_t\left[\left(\frac{M_{Ht+dt}}{M_{Ht}}-\frac{M_{Ft+dt}}{M_{Ft}}\frac{e_{t+dt}}{e_t}\right)\frac{B_{Ht+dt}}{B_{Ht}}\right]=0$, $E_t\left[\left(\frac{M_{Ht+dt}}{M_{Ht}}-\frac{M_{Ft+dt}}{M_{Ft}}\frac{e_{t+dt}}{e_t}\right)\frac{Y_{nt+dt}}{Y_{nt}}\right]=0$. Because the home bond return is conditionally deterministic, the first equation implies that $\frac{M_{Ht+dt}}{M_{Ht}}-\frac{M_{Ft+dt}}{M_{Ft}}\frac{e_{t+dt}}{e_t}$ is conditionally zero-mean, which then converts the last equation on Y_n into (3).

[5]In technical and general expression, risk entanglement avails in asset markets of $N+1$ traded basis assets $\{Y\}\equiv\{B_H,Y_n\}$, $n\in\{1,\ldots,N\}$ if $\exists Y\in\{Y\}$, and $i\in\mathcal{J}_\mathcal{Y}$ with $\Delta_{Yi}\neq 0$: $\nexists P\in\mathcal{L}\left(B_H,\{Y_n\}_{n=1}^N\right)$ s.t. $\frac{P_{t+dt}}{P_t}=1+\mu_P dt+\left(e^{\Delta_{Pi}}-1\right)\left(d\mathcal{N}_{it}-\lambda_i dt\right)$, where $\mathcal{J}_\mathcal{Y}$ denotes the set of jump types that impact the return on asset Y, and $P\in\mathcal{L}(\{B_H,Y_n\})$ a traded portfolio in the space spanned by $N+1$ basis assets. Further technical terms are given in Appendix A.

[6]See general Remark 1 below. Risk entanglement in discrete settings is examined in Appendix C.

exchange rate, an important step that we will also employ throughout the paper.

2.2 Complete Markets

Taking home investors' perspective, assume that there are three non-redundant basis assets, namely the home bond B_H and two risky assets Y_1 and Y_2. This is a complete-market setting because in continuous time, three basis assets $\{B_H, Y_1, Y_2\}$ that span the asset return space are sufficient to perfectly hedge both diffusion and jump risks impacting SDFs (1) and asset markets. Consequently, there exist traded portfolios that load only on a single risk. Without loss of generality, we hence specify the basis assets such that Y_1 loads only on the diffusion risk and Y_2 only on the jump risk. In the home currency denomination, basis asset returns are,

$$\frac{dB_{Ht}}{B_{Ht}} \equiv \frac{B_{Ht+dt}}{B_{Ht}} - 1 = r_H dt, \qquad \frac{dY_{1t}}{Y_{1t}} \equiv \frac{Y_{1t+dt}}{Y_{1t}} - 1 = \mu_{Y1} dt + \sigma_{Y1} dZ_t, \tag{4}$$

$$\frac{dY_{2t}}{Y_{2t}} \equiv \frac{Y_{2t+dt}}{Y_{2t}} - 1 = \mu_{Y2} dt + \left(e^{\Delta_{Y2}} - 1\right)\left(d\mathcal{N}_t - \lambda dt\right),$$

where μ_{Yn} is the conditional mean return of asset Y_n, σ_{Y1} the diffusion volatility of Y_1, and Δ_{Y2} the jump size of Y_2. By assumption, the foreign bond B_F is a traded asset, and therefore, its return denominated in the home currency is spanned as a portfolio of the basis returns,[7]

$$\frac{d\frac{B_{Ft}}{e_t}}{\frac{B_{Ft}}{e_t}} \equiv \frac{\frac{B_{Ft+dt}}{e_{t+dt}}}{\frac{B_{Ft}}{e_t}} - 1 = \left(1 - \sum_{n=1}^{2} \alpha_n\right) \frac{dB_{Ht}}{B_{Ht}} + \sum_{n=1}^{2} \alpha_n \frac{dY_{nt}}{Y_{nt}}, \tag{5}$$

where $\{\alpha_n\}$, $n \in \{1, 2\}$, are portfolio weights associated with assets $\{Y_n\}$ in the linear spanning. This spanning immediately implies a portfolio representation for the exchange rate,

$$\frac{e_{t+dt}}{e_t} = \frac{1 + \frac{dB_{Ft}}{B_{Ft}}}{1 + \left(1 - \sum_{n=1}^{2} \alpha_n\right) \frac{dB_{Ht}}{B_{Ht}} + \sum_{i=n}^{2} \alpha_n \frac{dY_{nt}}{Y_{nt}}}, \tag{6}$$

as well as the relationship between the diffusion and jump components of the exchange rate (2) and those of asset returns (4) (by applying Itô's lemma to match diffusion and jump terms of the

[7] The market integration assumption implies that every investor is able to trade all financial assets originating from home and foreign economies. Therefore, if basis returns $\left\{\frac{B_{Ht+dt}}{B_{Ht}}, \frac{Y_{1t+dt}}{Y_{1t}}, \frac{Y_{2t+dt}}{Y_t}\right\}$ span the space of traded returns denominated in the home currency, then basis returns $\left\{\frac{e_{t+dt}}{e_t} \frac{B_{Ht+dt}}{B_{Ht}}, \frac{e_{t+dt}}{e_t} \frac{Y_{1t+dt}}{Y_{1t}}, \frac{e_{t+dt}}{e_t} \frac{Y_{2t+dt}}{Y_t}\right\}$ span the space of traded returns denominated in the foreign currency, including foreign bond return $\frac{B_{Ft+dt}}{B_{Ft}}$. From this spanning follows the representation (5).

two sides of (5)),

$$\sigma_e = -\alpha_{Y1}\sigma_{Y1}, \qquad e^{\Delta_e} = \frac{1}{1+\alpha_2\left(e^{\Delta_{Y2}}-1\right)}. \tag{7}$$

The case in which the foreign bond is explicitly specified at the onset to be a basis asset (say Y_1) is a special case of (5) (with $\alpha_1 = 1$, $\alpha_2 = 0$). In such a case, effectively, the exchange rate is explicitly specified by (6), $\frac{e_{t+dt}}{e_t} = (1 + r_F dt)\frac{Y_{1t}}{Y_{1t+dt}}$, leaving no room for the exchange rate determination. The general case, therefore, does not specify the foreign bond at the onset. We instead solve for weights $\{\alpha_n\}$ and the exchange rate's diffusion and jump components (7), which then determine the associated exchange rate as well as the foreign bond specification ex-post via the representation (6). We discuss a consistent ex-post specification approach in details in Section 2.5 below.

In principle, it is important to observe a general market-based approach. The substitution of the exchange rate representation (6) into general cross-currency pricing equations (3) produces a system of two equations ($n \in 1, 2$) and two unknowns $\{\alpha_1, \alpha_2\}$, whose solutions determine the two respective exchange rates. In the current illustration of a complete market setting, this general solution approach reduces to a simple system of linear equations. Indeed, because each basis asset $\{Y_1, Y_2\}$ (4) loads only on one (diffusion or jump) risk, the pricing equations in (3) involve only one (diffusion or jump) risk parameter,

$$(\eta_F - \eta_H - \sigma_e)\,\sigma_{Y1} = 0, \qquad \lambda\left(e^{\Delta_H} - e^{\Delta_F+\Delta_e}\right)e^{\Delta_{Y2}} = 0$$

Note that $\sigma_{Y1} \neq 0$ and $\exp(\Delta_{Y2}) \neq 0$ for risky assets Y_1, Y_2, the substitution of exchange rate's diffusion and jump components (7) turns above equations into a linear equation system yielding a unique solution for weights $\{\alpha_1, \alpha_2\}$,

$$\begin{cases} \alpha_1\sigma_{Y1} = \eta_H - \eta_F \\ \\ \frac{1}{1+\alpha_2\left(e^{\Delta_{Y2}}-1\right)} = e^{\Delta_H-\Delta_F} \end{cases} \Longrightarrow \begin{cases} \alpha_1 = \frac{\eta_H-\eta_F}{\sigma_{Y1}} \\ \\ \alpha_2 = \frac{e^{\Delta_F-\Delta_H}-1}{e^{\Delta_{Y2}}-1}. \end{cases} \tag{8}$$

From these weights and the representation (6) follows a unique exchange rate solution. Reassuringly, the obtained exchange rate solution is equal to the ratio of SDFs, $e_t = \frac{M_{Ht}}{M_{Ft}}$, and hence, respects a well-known property of the complete market (e.g., Saa-Requejo (1994)).[8] We defer a full discussion of this exchange rate solution to Section 2.5 below, after examining alternative incomplete market

[8]In our current setting, the weights (8) and the associated exchange rate solution satisfy $\eta_e = \eta_H - \eta_F$, $e^{\Delta_e} = e^{\Delta_F-\Delta_H}$, which then indeed imply $e_t = \frac{M_{Ht}}{M_{Ft}}$.

settings.

2.3 Incomplete Markets: Risk Disentanglement

We now assume that there are two non-redundant basis assets, namely the home bond B_H and a risky asset Y_2 that loads only on the jump risk. This is an incomplete-market setting because in continuous time, two basis assets $\{B_H, Y_2\}$ that span the asset return space are insufficient to hedge the diffusion risk impacting SDFs (1).[9] Returns on the two basis assets $\{B_H, Y_2\}$ are given in the home currency denomination,

$$\frac{dB_{Ht}}{B_{Ht}} \equiv \frac{B_{Ht+dt}}{B_{Ht}} - 1 = r_H dt, \qquad \frac{dY_{2t}}{Y_{2t}} \equiv \frac{Y_{2t+dt}}{Y_{2t}} - 1 = \mu_{Y2} dt + \left(e^{\Delta_{Y2}} - 1\right)(d\mathcal{N}_t - \lambda dt), \tag{9}$$

Similar to (5), the assumed tradability of the foreign bond B_F is quantified by,

$$\frac{d\frac{B_{Ft}}{e_t}}{\frac{B_{Ft}}{e_t}} \equiv \frac{\frac{B_{Ft+dt}}{e_{t+dt}}}{\frac{B_{Ft}}{e_t}} - 1 = (1-\alpha_2)\frac{dB_{Ht}}{B_{Ht}} + \alpha_2 \frac{dY_{2t}}{Y_{2t}}. \tag{10}$$

This linear spanning implies a portfolio representation for the exchange rate and its jump component (in place of (6) and (7)),

$$\frac{e_{t+dt}}{e_t} = \frac{1 + \frac{dB_{Ft}}{B_{Ft}}}{1 + (1-\alpha_2)\frac{dB_{Ht}}{B_{Ht}} + \alpha_2 \frac{dY_{2t}}{Y_{2t}}}, \qquad e^{\Delta_e} = \frac{1}{1 + \alpha_2\left(e^{\Delta_{Y2}} - 1\right)}. \tag{11}$$

The substitution of the exchange rate representation (11) into general cross-currency pricing equations (3) produces a single linear equation (of a single unknown α_2) in the current incomplete market setting,

$$\lambda\left(e^{\Delta_H} - e^{\Delta_F + \Delta_e}\right) e^{\Delta_{Y2}} = 0 \Longrightarrow e^{\Delta_H} - e^{\Delta_F + \Delta_e} = 0 \Longrightarrow \alpha_2 = \frac{e^{\Delta_F - \Delta_H} - 1}{e^{\Delta_{Y2}} - 1}, \tag{12}$$

where the last equation arises from the substitution of the exchange rate volatility in (11). From this weight solution and the representation (11) follows a unique exchange rate solution. Interestingly, the obtained exchange rate solution is equal to the ratio of SDFs projected on spaces of asset returns denominated in respective currency, $e_t = \frac{M_{H\|t}}{M_{F\|t}}$, where projectors are $M_{H\|t} = Proj(M_{Ht}|\{B_{Ht}, Y_{2t}\})$, $M_{F\|t} = Proj(M_{Ft}|\{B_{Ht}e_t, Y_{2t}e_t\})$ (see Appendix D for a discussion on

[9] In standard equilibrium asset pricing setting, SDF proxies for the marginal utility of consumption. Shocks to the SDF are shocks to the marginal utility, and are risks to economic agents.

SDF projectors). This equality affirms a relationship frequently employed in the determination of the exchange rate in the incomplete market literature (e.g., Brandt et al. (2006)).[10]

We remark that in place of the incomplete market considered above (i.e., basis assets $\{B_H, Y_2\}$ (9) span only the jump risk) a similar conclusion is achieved in the alternative incomplete market setting in which traded assets only span the diffusion risk (i.e., when basis assets are $\{B_H, Y_1\}$, a subset of specification (4)).[11] Both of these incomplete market settings share a common important property that every risk impacting asset markets (not risks impacting SDFs or the general economies) is singly contracted by traded assets (or portfolio of traded assets). We refer to this property as risks being completely disentangled in asset markets. We defer a full discussion of the risk disentanglement to Section 2.5 below, after examining a deeper concept of risk entanglement.

2.4 Incomplete Markets: Risk Entanglement

We now assume that there are two non-redundant basis assets, namely the home bond B_H and a risky asset Y_3 that loads on both diffusion and jump risks. This is an incomplete-market setting because in continuous time, two basis assets $\{B_H, Y_3\}$ that span the asset return space are insufficient to perfectly hedge diffusion and jump risks impacting SDFs (1) and asset markets. In the home currency denomination, returns on the two basis assets are given,

$$\frac{dB_{Ht}}{B_{Ht}} \equiv \frac{B_{Ht+dt}}{B_{Ht}} - 1 = r_H dt, \qquad \frac{dY_{3t}}{Y_{3t}} \equiv \frac{Y_{3t+dt}}{Y_{3t}} - 1 = \mu_3 dt + \sigma_3 dZ_t + \left(e^{\Delta_{Y3}} - 1\right)(d\mathcal{N}_t - \lambda dt)\,. \quad (13)$$

Similar to (5) and (10), the assumed tradability of the foreign bond B_F is quantified by,

$$\frac{d\frac{B_{Ft}}{e_t}}{\frac{B_{Ft}}{e_t}} \equiv \frac{\frac{B_{Ft+dt}}{e_{t+dt}}}{\frac{B_{Ft}}{e_t}} - 1 = (1-\alpha_3)\frac{dB_{Ht}}{B_{Ht}} + \alpha_3 \frac{dY_{3t}}{Y_{3t}}. \quad (14)$$

[10] We prove this result in the general setting in Appendix D. Notation $M_{H\|t}$ (and $M_{F\|t}$) denotes the home (and foreign) SDF projected on the asset return space in the home (and foreign) currency denomination.

[11] Specifically, when basis assets are $\{B_H, Y_1\}$, in place of (11), the exchange rate portfolio representation is $\frac{e_{t+dt}}{e_t} = \frac{1+\frac{dB_{Ft}}{B_{Ft}}}{1+(1-\alpha_2)\frac{dB_{Ht}}{B_{Ht}}+\alpha_1\frac{dY_{1t}}{Y_{1t}}}$, which implies an exchange rate diffusion $\sigma_e = -\alpha_1\sigma_{Y1}$. In place of (12), now (3) produces a single linear equation (of a single unknown α_1), $(\eta_F - \eta_H - \sigma_e)\,\sigma_{Y1} = 0$, or equivalently, $\eta_F - \eta_H + \alpha_1\sigma_{Y1} = 0$. A single solution for the weight $\sigma_{Y1} = \frac{\eta_H - \eta_F}{\sigma_{Y1}}$ then determines a single possible exchange rate via the above portfolio representation. This exchange rate solution again satisfies the relationship $e_t = \frac{M_{H\|t}}{M_{F\|t}}$, where projectors are in return spaces of the respective currencies, $M_{H\|t} = Proj(M_{Ht}|\{B_{Ht}, Y_{1t}\})$, $M_{F\|t} = Proj(M_{Ft}|\{B_{Ht}e_t, Y_{1t}e_t\})$.

This linear spanning implies a portfolio representation for the exchange rate, and its matching diffusion and jump components,

$$\frac{e_{t+dt}}{e_t} = \frac{1 + \frac{dB_{Ft}}{B_{Ft}}}{1 + (1-\alpha_3)\frac{dB_{Ht}}{B_{Ht}} + \alpha_3 \frac{dY_{3t}}{Y_{3t}}}, \qquad \sigma_e = -\alpha_3\sigma_3, \qquad e^{\Delta_e} = \frac{1}{1+\alpha_3\left(e^{\Delta_{Y3}}-1\right)}. \tag{15}$$

The substitution of the exchange rate representation (15) into general cross-currency pricing equations (3) produces,

$$(\eta_F - \eta_H - \sigma_e)\,\sigma_{Y3} + \lambda\left(e^{\Delta_H} - e^{\Delta_F+\Delta_e}\right)e^{\Delta_{Y3}} = 0,$$

or equivalently (by employing exchange rate's diffusion and jump components in (15)),

$$(\eta_F - \eta_H + \alpha_3\sigma_{Y3})\,\sigma_{Y3} + \lambda\left(e^{\Delta_H} - \frac{e^{\Delta_F}}{1+\alpha_3\left(e^{\Delta_{Y3}}-1\right)}\right)e^{\Delta_{Y3}} = 0. \tag{16}$$

Clearly, this is a quadratic equation of weight α_3. This non-linearity is a remarkable deviation from the linear equations (8) in complete markets and (12) in incomplete markets with disentangled risks. The non-linearity indicates potentially multiple pricing-consistent exchange rates given the same inputs of SDFs M_H, M_F (1) and asset returns specified in the home currency denomination $\{B_H, Y_3\}$. This novel possibility arises precisely because in the current incomplete market setting, diffusion and jump risks are always mixed in asset returns spanned by basis assets $\{B_H, Y_3\}$ (13). We will refer to this important asset market feature as risk entanglement to distinguish it from settings of complete markets or incomplete markets with disentangled risks considered earlier. We defer a full discussion of the risk entanglement to Section 2.5 below, after examining the possible multiplicity of exchange rate solutions and its modeling aspects in international finance.

Technically, the non-linearity of equation (16) depends crucially on the significance of the jump risks in asset returns. In fact, when the jump size on Y_3's return is sufficiently small, equation (16) becomes linear in unknown α_3,

$$\left|e^{\Delta_{Y3}}-1\right| \ll 1 \Longrightarrow (\eta_F - \eta_H + \alpha_3\sigma_{Y3})\,\sigma_{Y3} + \lambda\left(e^{\Delta_H} - e^{\Delta_F}\left[1-\alpha_3\left(e^{\Delta_{Y3}}-1\right)\right]\right)e^{\Delta_{Y3}} = 0, \tag{17}$$

effectively yielding a single exchange rate solution.[12] Hence, risk entanglement technically requires significant jump risks. Empirically, jumps of significant sizes induce skewed distributions, which are an important feature of FX markets as documented in Brunnermeier et al. (2008), Burnside

[12]This is also the reason why incomplete asset markets featuring only diffusion risks always belong to the class of complete risk disentanglement settings (see Remark 1).

et al. (2011), and Farhi et al. (2015).

In general (when jump risks are significant), the quadratic equation (16) possesses two solutions $\alpha_3^{\pm}$, giving rise to two corresponding pricing-consistent exchange rate solutions $e_t^{\pm}$ (15) ex-post,[13]

$$\frac{e_{t+dt}^{\pm}}{e_t^{\pm}} = \frac{1 + \frac{dB_{Ft}}{B_{Ft}}}{1 + \left(1 - \alpha_3^{\pm}\right) \frac{dB_{Ht}}{B_{Ht}} + \alpha_3^{\pm} \frac{dY_{3t}}{Y_{3t}}}. \tag{18}$$

While both of these exchange rates are consistent with the same input specifications $\{M_H, M_F, B_H, Y_3\}$ of the setup, it is crucial to observe that they pertain to two different economies. These economies are associated with different asset returns in the foreign currency denomination, namely either $\{e^+ B_H, e^+ Y_3\}$ or $\{e^- B_H, e^- Y_3\}$. In particular, by virtue of the portfolio representation (14), the functional specification of the foreign bond (in terms of asset returns in the home currency denomination) is distinct in the two economies,

$$\frac{B_{Ft+dt}^{+}}{B_{Ft}^{+}} \frac{e_t^{+}}{e_{t+dt}^{+}} = 1 + \left(1 - \alpha_3^{+}\right) \frac{dB_{Ht}}{B_{Ht}} + \alpha_3^{+} \frac{dY_{3t}}{Y_{3t}}, \tag{19}$$

$$\frac{B_{Ft+dt}^{-}}{B_{Ft}^{-}} \frac{e_t^{-}}{e_{t+dt}^{-}} = 1 + \left(1 - \alpha_3^{-}\right) \frac{dB_{Ht}}{B_{Ht}} + \alpha_3^{-} \frac{dY_{3t}}{Y_{3t}}. \tag{20}$$

If one starts out with specifications of asset returns also in the foreign currency denomination that are associated with a particular exchange rate solution, then we will consistently and uniquely recover that same underlying exchange rate ex-post. E.g., if at the onset we substitute the specification of an original risky basis asset (i.e., asset Y_3 in the current illustration) by the specification (19) of the foreign bond return in the home currency denomination (i.e., B_F now is specified explicitly as a basis asset),[14] then the unique consistent exchange rate is e^+. Vice versa, the specification (20) at the onset consistently implies the unique exchange rate e^-. By construction, both choices (19) and (20) are consistent with the original input specifications $\{M_H, M_F, B_H, Y_3\}$. Therefore, the exchange rate multiplicity in presence of risk entanglement is different from the notion of multiple equilibria in economic dynamics and does not entail the nuisance of ad-hoc stabilization of one equilibrium over the others. We discuss this modeling advantage of risk entanglement in further details below.

[13]Assuming regularity conditions, quadratic equation (16) has two real solutions $\alpha_3^{\pm} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, where $a = \sigma_{Y3}^2 \left(e^{\Delta_{Y3}} - 1\right)$, $b = \sigma_{Y3}^2 + \sigma_{Y3}\left(\eta_F - \eta_H\right)\left(e^{\Delta_{Y3}} - 1\right) + \lambda e^{\Delta_H} e^{\Delta_{Y3}} \left(e^{\Delta_{Y3}} - 1\right)$, $c = \sigma_{Y3}\left(\eta_F - \eta_H\right) + \lambda e^{\Delta_{Y3}} \left(e^{\Delta_H} - e^{\Delta_F}\right)$.

[14]Such a substitution of two specifications assures that basis assets remain non-redundant because the number of specified basis assets remain the same.

2.5 Discussion

Key aspects of how entangled risks in asset markets determine the set of possible exchange rates can be intuitively elucidated in our various illustrations in Sections 2.2-2.4 above. Our discussion in the current section aims to contrast the entanglement with disentanglement of asset market risks, and build intuitions for general results derived in later sections. Several important observations are in order.

First, the exchange rate determination setup in all illustrations commonly relies on an explicit formulation of the foreign bond tradability assumption in terms of basis assets as well as the no-arbitrage consistent pricing across currencies. We start with given home and foreign SDFs and basis asset returns specified in the home currency denomination. While the assumed tradability of the foreign bond establishes a portfolio representation for the exchange rate ((6), (11), or (15)), the cross-currency pricing (3) turns this representation into a balanced equation system ((8), (12), or (16)) of portfolio weights to determine all possible pricing-consistent exchange rates.[15] While our approach does not explicitly model structural features underlying SDF specifications M_H and M_F, equilibrium exchange rates of structural models (that preserve no-arbitrage and market integration) respect pricing consistency, and thus, necessarily obey system (3). This paper's joint analysis of SDFs, asset returns, and exchange rates helps to exhaustively map their possible combinations that respect pricing consistency. If a combination is found in which (i) SDFs specified as inputs have certain desirable characteristics and (ii) an exchange rate solution obtained as an output matches its desirable moments in the data, then our analysis informs structural models in adopting such SDF characteristics (and the associated asset market configuration) so that the desirable exchange rate consistently emerges in equilibrium.

Second, asset-risk configuration, i.e., how risks are embedded in available traded assets, is crucial in the determination of possible exchange rate solutions and their dynamics. When every risk impacting asset markets is individually contracted, the equations in the system (3) that determine pricing-consistent exchange rates become decoupled: the equation associated with basis asset Y_n concerns only a single unknown weight α_n of the exchange rate representation portfolio. As a result, each α_n is obtained uniquely by solving a separate pricing equation, yielding a unique consistent exchange rate $e_t = \frac{M_{Ht}}{M_{Ft}}$. We refer to this asset-risk configuration as risks being completely disentangled in asset markets (Definition 1). By construction, the complete market illustration of

[15] We observe that every risky basis asset is associated with (i) one non-redundant equation in (3) and (ii) one weight in the exchange rate portfolio representation (e.g., (6)). As a result, the equation system is balanced.

Section 2.2 always features completely disentangled risks (4), resulting in the completely decoupled equation system (8) that solves for a unique exchange rate. Asset returns in the incomplete market illustration of Section 2.3 are impacted only by a jump risk, which is individually contracted by traded asset Y_2 (9). That illustration hence also belongs to the class of completely disentangled risks, yielding the completely decoupled equation system (12) and a unique consistent exchange rate.

When some jump risks impacting asset markets cannot be individually contracted in asset markets, the equation system (3) is coupled: the pricing equation associated with (some) basis asset Y_n always concerns several risks that jointly impact Y_n. While the exchange rate portfolio representation turns the system (3) into equations of unknown weights α_n, the jump risk nature makes these equations nonlinear. As a result, there exist multiple solutions of weights $\{\alpha_n\}$, and multiple corresponding pricing-consistent exchange rates. We refer to this market setting as risks being entangled in asset markets, i.e., the risk-asset embedding in which many distinct risks are contracted in few available traded assets (see the formal Definition 1 above). The incomplete market illustration $\{B_H, Y_3\}$ (13) of Section 2.4 features diffusion and jump risks, both impact the basis asset Y_3 (and none impacts B_H). Therefore, every traded asset spanned by basis assets $\{B_H, Y_3\}$ either features both diffusion and jump risks, or no risks at all. That is, diffusion and jump risks are entangled in the asset markets (13). That illustration hence also belongs to the class of entangled risks, yielding a nonlinear equation (16) that solves for two consistent exchange rate $e^{\pm}$ (18).

Third, multiple solutions of the exchange rate in the risk entanglement setting are not different equilibria of some single underlying structural model of the exchange rate. This is because each exchange rate solution identifies and corresponds to a different foreign asset return specification, and thus, a different market economy ex-post. If one moves from one solution of the exchange rate to another, one moves from one market setting to another, in which home investors perceive the foreign assets differently as we elaborate at the end of Section 2.4. This ex-ante exchange rate multiplicity offers a modeling advantage of the risk entanglement approach. Indeed, obtaining multiple exchange rates (i) widens the exchange rate possibilities to match the the data, and (ii) identifies the ex-post consistent asset market which gives rise to each corresponding ex-ante exchange rate solution. To illustrate, having achieved two pricing-consistent exchange rate solutions $e^{\pm}$ (18), we may focus on the desirable one that fits the empirical counterpart better (e.g., matching exchange rate moments in data), say e^{-}. This exchange rate solution then unambiguously identifies the corresponding foreign bond return specification (20) in the home currency, that if we start with such a specification (in place of the specification of an original risky basis asset), the unique consistent exchange rate is the desirable e^{-}. This ex-post specification approach highlights the fact one does not need to explicitly specify the foreign bond as a basis asset at the onset per the discussion following equation (7).[16]

Finally, none of the multiple exchange rate solutions that arise in the presence of risk entanglement is equal to the ratio of countries' SDF projectors. The deviation from a linear relationship between the exchange rate and SDF dynamics enables the risk entanglement mechanism to address international finance puzzles. Intuitively, a higher degree of risk entanglement in asset markets leads to a higher degree of non-linearity in the equation system determining the exchange rate. As a result, exchange rate solutions can deviate further from SDF dynamics. The incomplete market illustration of Section 2.4 features entangled risks, yielding two exchange rate solutions, none of which equals the respective ratio of SDF projectors, $e_t \neq \frac{M_{H\|t}}{M_{F\|t}}$ (with $M_{H\|t} = Proj(M_{Ht}|\{B_{Ht}, Y_{3t}\})$, $M_{F\|t} = Proj(M_{Ft}|\{B_{Ht}e_t, Y_{3t}e_t\})$). When more risk types are added to the setting without adding more traded assets (i.e., risks are more entangled), quadratic equation (16) becomes a higher-degree polynomial equation, yielding more exchange rate solutions of richer dynamics. Next, we briefly present formal risk entanglement results that generalize the findings from the above illustrations.

2.6 Risk Entanglement: General Case

We formalize the exchange rate determination setup and state general risk entanglement results for arbitrary risk-asset embeddings in FX markets in a generic jump diffusion framework. Underlying intuitions of the general results are informed by the discussion above. We relegate details on the notations and derivations to Appendix A, in the interest of space. We first recapitulate two customary but important assumptions (apart from no-arbitrage).

Assumption 1 *We assume that (i) international asset markets are fully integrated international asset markets and (ii) short-term risk-free bonds of every country are traded in international asset markets.*

The market setup to determine pricing-consistent exchange rates is as follows.

Protocol 1 (No-arbitrage Determination of the Exchange Rate)

[16]That is, the consistency is not compromised by not explicitly specifying the foreign bond as a basis asset at the onset.

Step 1: *We first take as given, (i) the gross return processes of basis assets* $\left\{\frac{B_{Ht+dt}}{B_{Ht}}, \frac{Y_{nt+dt}}{Y_{nt}}\right\}$, $n \in \{1, \ldots, N\}$, *specified in the home currency denomination. All other traded assets are portfolios of these basis assets,*[17] *and (ii) distinctly specified SDFs* M_{Ht}, M_{Ft} *of countries involved.*

Step 2: *We then determine the exchange rate process* e_t *based on the requirements that, (a) it prices traded assets consistently across currency denominations, and (b) a country's risk-free bond remains a traded asset to the other country's investors (after the bond's payoff is converted to the other currency).*

Step 3: *Ex-post, each specific exchange rate solution* e_t *determined above identifies a distinct corresponding market specification (including the foreign bond return in the home currency denomination) that if we start with such a market specification ex-ante, the only consistent exchange rate is the specific solution* e_t *under consideration.*

The market approach to quantify the exchange rate has been explored in the finance literature at least since Saa-Requejo (1994). Our innovations are to introduce a portfolio representation into this approach to analytically characterize exchange rate solutions (in Step 2), and identify a corresponding ex-ante market configuration that gives rise uniquely to each of the exchange rate solutions ex-post (in Step 3). To incorporate structural features, SDF specifications M_{Ht}, M_{Ft} in Protocol 1 may be taken as desirable SDFs suggested in the structural literature.

Specifically, assumption 1 on bonds' tradability and market integration assures that the foreign bond return in the home currency is spanned by basis returns denominated in the home currency. This spanning gives rise to a portfolio representation for the exchange rate, generalizing (6), (11), and (15),

$$\frac{e_t}{e_{t+dt}} = \frac{1}{1 + r_F dt} \left\{ \left(1 - \sum_{Y \in \{Y\}} \alpha_Y \right) \frac{B_{H,t+dt}}{B_{H,t}} + \sum_{Y \in \{Y\}} \alpha_Y \frac{Y_{t+dt}}{Y_t} \right\}. \tag{21}$$

The substitution of this representation into the cross-currency pricing consistency condition (3) generates a balanced system of N unknown weights $\{\alpha_n\}$ and N equations (one equation per risky basis asset, see (30), Appendix A). Given this setup, we state the main risk entanglement result in the determination of the exchange rate.

[17] The choice of denomination currency is non-material, so we conventionally choose it to be the home currency. In this convention, $N + 1$ basis assets consist of the home bond, and N risky assets $\{Y_n\}$, $n \in \{1, \ldots, N\}$.

Theorem 1 *1. Assuming markets' perfect integration and risk-free bonds' tradability (Assumption 1), the exchange rate determination of Protocol 1 is feasible when the associated weights* $\{\alpha_Y\}$ *in* (21) *solve the system of pricing equations* (3) *applied on all traded assets* Y*. Furthermore, every exchange rate determined by this approach is pricing-consistent.*[18]

2. This equation system (3) *(or equivalently,* (30)*) is non linear, and hence has multiple solutions (a unique solution)* $\{\alpha_Y\}$ *if and only if risks in asset markets are entangled (completely disentangled).*[19] *Accordingly, in the presence of risk entanglement (complete risk disentanglement), there exist multiple pricing-consistent exchange rates (a unique exchange rate)* e_t *determined along Protocol 1.*

In essence, risk entanglement is the only possible market-based ingredient to make the relationship between the exchange rate and country-specific pricing dynamics ambiguous by mapping multiple exchange rate solutions into a given pair of country-specific SDFs. Intuitively, in "thin" financial markets (i.e., multiple risks entangled in few traded assets), price data (observed in all currencies) is insufficient to unambiguously infer prices of individual risks in any currency, giving rise to multiple pricing-consistent exchange rates. Exchange rate solutions depend not only on the given SDFs, but also crucially on the specificity in which risks are contracted in traded asset returns. This risk-asset embedding is a new market-based approach to rationalize the exchange rate and international pricing dynamics.

We remark two special cases of complete risk disentanglement (Definition 1): (i) complete markets and (ii) incomplete markets with pure-diffusion risks. While the first case is evident, the second case arises because we can always linearly combine assets (and redefine/rotate diffusion risks) into portfolios that load on single redefined diffusion risks.[20] We summarize this observation in the following remark.

Remark 1 *Diffusion risks are completely disentangled in any (incomplete) asset market.*

[18]In the context of Protocol 1, a pricing-consistent exchange rate respects Assumption 1: (i) the given foreign SDF M_F consistently prices all traded assets when they are denominated in the foreign currency, and (ii) respects the tradability of all bonds by upholding the portfolio representation (21) of the exchange rate. That is, the exchange rate's drift, diffusion and jump components (28)-(29) satisfy the equation system (30).

[19]Technically, the statement "if and only if" in Theorem 1 applies as long as the equation system (3) (or more explicitly, (30)) remains genuinely non-linear. This specification rules out a special set of parametric values, under which the equation system reduces to a degenerate non-linear system (which possess a single, but special, solution).

[20]To illustrate, suppose d independent diffusion risks impact $N < d$ non-redundant asset returns. We can always redefine (i.e., linearly combine) and partition the d original diffusion risks into two orthogonal subsets; N diffusion risks are completely disentangled by N asset returns, and the remaining $(d-N)$ diffusion risks drop out because they are uncorrelated with, and not detected by, asset returns. This asset market setting is hence completely disentangled per Definition 1.

The pricing-consistent exchange rate is known to be unique in the literature in two special cases of complete risk disentanglement, namely (i) complete markets (e.g., Saa-Requejo (1994)), and (ii) incomplete markets with pure-diffusion risks (e.g., Brandt et al. (2006)). Reassuringly, Remark 1 indicates their complete risk disentanglement nature, and Theorem 1 reaffirms these special findings of the literature.

3 International Finance Puzzles

In this section, we calibrate a model featuring risk entanglement to address (i) the international correlation puzzle (co-existence of smooth exchange rates and mildly correlating SDFs), (ii) sizable currency excess returns, and (iii) low correlations between growths of the exchange rate and SDF ratio.[21] The numerical setup follows our Protocol 1, in which all input SDFs are the given full SDFs (but not their projectors). We further contrast these results with an example of complete markets (and completely disentangled risks), in which the exchange rate is always equal to the ratio of country-specific SDFs, and therefore, international correlations are deemed puzzling. Conceptually, as we discuss below Theorem 1 and also Appendix A, risk entanglement (as an instance of incomplete market settings) not only decouples the exchange rate from the ratio of SDFs, but also enables the level of decoupling to depend on the risk-asset configuration in markets. Our numerical examples in this section regulate this risk-asset configuration to illustrate risk entanglement at work.

We contrast two economies: (I) an economy with complete risk disentanglement, and (II) an economy with entangled jump and diffusion risks. In economy (I) with completely disentangled risks we assume there are two diffusion processes dZ_{1t} and dZ_{2t}. In economy (II) with risk entanglement we introduce a single Poisson jump process $d\mathcal{N}_{1t}$ in addition to the two diffusion risks. Hence, the difference between the two economies stems from the additional jump risk in economy (II). SDFs M_H and M_F are exposed to all risk sources in each respective economy,

$$\frac{dM_{It}}{M_{It}} = -r_I dt - \eta_{I1} dZ_{1t} - \eta_{I2} dZ_{2t} + \left(e^{\Delta_{I1}} - 1\right)\left(d\mathcal{N}_{1t} - \lambda_1 dt\right), \qquad I \in \{H, F\},$$

where in economy (I) with completely disentangled risks we set $\Delta_{I1} = 0$, $\forall I \in \{H, F\}$. In both

[21] The last pattern is inspired by and adopted from the Backus and Smith (1993) puzzle (i.e., low empirical correlations between growths of the exchange rate and consumption ratio) for pure market-based settings, wherein SDFs are some monotone functions of equilibrium consumptions.

economies we assume that home investors can trade one risk-free bond B_H and three risky assets Y_1, Y_2 and the foreign bond (which is risky when denominated in the home currency, i.e., $\frac{B_F}{e}$). Notice that only two of these three risky assets are non-redundant according to the portfolio representation of the foreign bond (21). Foreign investors trade the same assets. For simplicity we assume that risks impacting the economy are also in the traded asset space,

$$\begin{aligned}
\frac{dY_{jt}}{Y_{jt}} &= \mu_{Y_j} dt + \sigma_{Y_j 1} dZ_{1t} + \sigma_{Y_j 2} dZ_{2t} + \left(e^{\Delta_{Y_j 1}} - 1\right)(d\mathcal{N}_{1t} - \lambda_1 dt), \qquad j \in \{1, 2\} \\
\frac{dB_{Ht}}{B_{Ht}} &= r_H dt,
\end{aligned}$$

where we set $\Delta_{Y_j 1} = 0$, $\forall j \in \{1, 2\}$, in economy (I), as a special case of the above asset returns. Therefore, economy (I) is a complete market economy (2 diffusion risks and 2 non-redundant risky assets and a risk-free bond), and thus features completely disentangled risks. Assuming additional risks which affect the SDFs but not the traded assets is a straightforward extension but does not conceptually change our numerical illustration.

Tables 1 and 2 report the initially fixed SDFs and asset space (denominated in the home currency) and one corresponding pricing consistent exchange rate solution for each economy (I) and (II). The pricing consistent exchange rate is solved by employing the portfolio approach outlined in Section 2.6 and Appendix A. We substitute the parameters specifying the SDFs and the two risky assets in Tables 1 and 2 into the system of two equations (30) and solve for the two unknown portfolio weights α_1 and α_2. Then, we plug α_1 and α_2 into the portfolio representation (21) to obtain the pricing consistent exchange rate. In the case of completely disentangled risks (economy (I)) the solution to (30) and thus the exchange rate is unique according to Theorem 1. In contrast, if risks are entangled (economy (II)) we get multiple solutions for the portfolio weights α_1 and α_2 satisfying (30) and thus multiple exchange rates which are pricing consistent with the given SDFs and asset space (Theorem 1). In particular, in the case of a single jump type, two-dimensional diffusion and two non-redundant risky assets, the system (30) boils down to two quadratic equations. We pick the solution which appears economically most reasonable, i.e., the exchange rate which matches the first and second moments of the data the best.[22]

Given the risk loadings of the exchange rate we calculate the total exchange rate volatility

[22]In our numerical example the unreported solutions produce exchange rates which clearly do not match the data well and it is obvious which solution is more sensible. In general, with more complicated entanglement of many risks, several exchange rate solutions might fit the data reasonably well.

(which includes contributions from both diffusion and jump risks),

$$Vol_t\left(\frac{de_t}{e_t}\right)=\sqrt{\sum_{i=1}^{2}\sigma_{ei}^2dt+\lambda_1 dt\left(e^{\Delta_{e1}}-1\right)^2}, \tag{22}$$

the carry trade premium to the home investor of borrowing the foreign currency and lending the home currency,

$$E_t\left[CT^{H}_{-F/+H}\right]=\sigma_{e1}\eta_{H1}+\sigma_{e2}\eta_{H2}+\lambda_1\left(e^{\Delta_{H1}}-1\right)\left(e^{-\Delta_{e1}}-1\right), \tag{23}$$

and the total correlation between the exchange rate growth and the growth in the ratio of home to foreign SDFs,

$$Corr_t\left(\frac{d\left(\frac{M_{Ht}}{M_{Ft}}\right)}{\frac{M_{Ht}}{M_{Ft}}},\frac{de_t}{e_t}\right)=\frac{Cov_t\left(\frac{d\left(\frac{M_{Ht}}{M_{Ft}}\right)}{\frac{M_{Ht}}{M_{Ft}}},\frac{de_t}{e_t}\right)}{Vol_t\left(\frac{d\left(\frac{M_{Ht}}{M_{Ft}}\right)}{\frac{M_{Ht}}{M_{Ft}}}\right)Vol_t\left(\frac{de_t}{e_t}\right)}, \tag{24}$$

with

$$\begin{aligned}Cov_t\left(\frac{d\left(\frac{M_{Ht}}{M_{Ft}}\right)}{\frac{M_{Ht}}{M_{Ft}}},\frac{de_t}{e_t}\right) &= \sum_{i=1}^{2}\left(\eta_{Fi}-\eta_{Hi}\right)\sigma_{ei}dt+\lambda_1 dt\left(e^{\Delta_{H1}-\Delta_{F1}}-1\right)\left(e^{\Delta_{e1}}-1\right)\\ Vol_t\left(\frac{d\left(\frac{M_{Ht}}{M_{Ft}}\right)}{\frac{M_{Ht}}{M_{Ft}}}\right) &= \sqrt{\sum_{i=1}^{2}\left(\eta_{Fi}-\eta_{Hi}\right)^2dt+\lambda_1 dt\left(e^{\Delta_{H1}-\Delta_{F1}}-1\right)^2}.\end{aligned}$$

Table 1 summarizes the results in the diffusion setting with completely disentangled risks (economy (I)). Table 2 contains the values in the jump-diffusion setting with entangled risks (economy (II)). We choose the market prices η_{I1}, η_{I2} and Δ_{I1} $\forall I\in\{H,F\}$ such that the total volatilities of M_H and M_F are identical and roughly 60%, and the total correlation between the two SDF growths is 30%.[23] The large SDF volatilities are consistent with the Hansen and Jagannathan (1991) bound, and the modest SDF correlation matches the correlation between consumption growths across developed economies (Brandt et al., 2006). Jump sizes in the two SDFs (in the jump-diffusion economy (II)) are symmetric, $\Delta_{H1}=\Delta_{F1}=4\%$. We interpret Y_1 as the stock market (denominated in the home currency). Therefore, we choose the diffusion risk loadings $\sigma_{Y_1 1}$ and $\sigma_{Y_1 2}$ such that the

[23] All variances, covariances and correlations in this section are total variances, covariances and correlations, i.e., they include diffusion and jump risks, see (22), (23), (24).

Table 1: Exchange Rate in Economy with Completely Disentangled Risk

SDFs M_H,M_F: $\frac{dM_{It}}{M_{It}} = -r_I dt - \eta_{I1} dZ_{1t} - \eta_{I2} dZ_{2t}$

Risky Assets Y_1,Y_2: $\frac{dY_{jt}}{Y_{jt}} = \mu_{Y_j} dt + \sigma_{Y_j 1} dZ_{1t} + \sigma_{Y_j 2} dZ_{2t}$

	SDFs and Asset Space			
	M_H	M_F	Y_1	Y_2
Diffusion dZ_{1t}	$\eta_{H1} = 0.04$	$\eta_{F1} = 0.60$	$\sigma_{Y_1 1} = 0.109$	$\sigma_{Y_2 1} = 0.110$
Diffusion dZ_{2t}	$\eta_{H2} = 0.62$	$\eta_{F2} = 0.15$	$\sigma_{Y_1 2} = 0.104$	$\sigma_{Y_2 2} = 0.102$
Volatility	62%	62%	15%	15%
Risk Premium	NA	NA	6.8%	6.8%

	Exchange Rate e
Diffusion dZ_{1t}	$\sigma_{e1} = 0.56$
Diffusion dZ_{2t}	$\sigma_{e2} = -0.47$
Volatility	73.2%
Carry Trade Premium	26.8%
$Corr_t\left(\frac{dM_H}{M_H}, \frac{dM_F}{M_F}\right)$	30%
$Corr_t\left(\frac{d\left(\frac{M_{Ht}}{M_{Ft}}\right)}{\frac{M_{Ht}}{M_{Ft}}}, \frac{de_t}{e_t}\right)$	100%

Notes: Given the quantities specifying the market prices of risk (risk loadings of SDFs M_H and M_F) and the risk exposures of the two traded assets Y_1 and Y_2, we determine the pricing-consistent exchange rate according to the system of two equations (30). In the case of economy (II) with entangled risks where multiple pricing-consistent exchange rates exist, we only report the one solution which best matches the first two moments of exchange rates in the data. $Corr_t\left(\frac{dM_H}{M_H}, \frac{dM_F}{M_F}\right) = \frac{Cov_t\left(\frac{dM_H}{M_H}, \frac{dM_F}{M_F}\right)}{Vol_t\left(\frac{dM_H}{M_H}\right) Vol_t\left(\frac{dM_F}{M_F}\right)}$ is the correlation between the two SDFs, where $Cov_t\left(\frac{dM_H}{M_H}, \frac{dM_F}{M_F}\right) = \sum_{i=1}^{2} \eta_{Hi}\eta_{Fi} dt$ is the covariance between the two SDFs and $Vol_t\left(\frac{dM_I}{M_I}\right) = \sqrt{\sum_{i=1}^{2} \eta_{Ii}^2 dt}$ is the volatility of SDF I. The formulas to compute the total exchange rate volatility, carry trade premium and the correlation between the exchange rate growth and the growth in the ration of SDFs $\left(Corr_t\left(\frac{d\left(\frac{M_{Ht}}{M_{Ft}}\right)}{\frac{M_{Ht}}{M_{Ft}}}, \frac{de_t}{e_t}\right)\right)$ are given in (22), (23), (24).

Table 2: Exchange Rate in Economy with Entangled Risk

SDFs M_H,M_F: $\frac{dM_{It}}{M_{It}} = -r_I dt - \eta_{I1} dZ_{1t} - \eta_{I2} dZ_{2t} + \left(e^{\Delta_{I1}} - 1\right)(d\mathcal{N}_{1t} - \lambda_1 dt)$

Risky Assets Y_1,Y_2: $\frac{dY_{jt}}{Y_{jt}} = \mu_{Y_j} dt + \sigma_{Y_j 1} dZ_{1t} + \sigma_{Y_j 2} dZ_{2t} + \left(e^{\Delta_{Y_j 1}} - 1\right)(d\mathcal{N}_{1t} - \lambda_1 dt)$

	SDFs and Asset Space			
	M_H	M_F	Y_1	Y_2
Diffusion dZ_{1t}	$\eta_{H1} = 0.04$	$\eta_{F1} = 0.60$	$\sigma_{Y_1 1} = 0.109$	$\sigma_{Y_2 1} = 0.11$
Diffusion dZ_{2t}	$\eta_{H2} = 0.62$	$\eta_{F2} = 0.15$	$\sigma_{Y_1 2} = 0.104$	$\sigma_{Y_2 2} = 0.102$
Jump $d\mathcal{N}_{1t}$	$\Delta_{H1} = 0.04$	$\Delta_{F1} = 0.04$	$\Delta_{Y_1 1} = -0.025$	$\Delta_{Y_2 1} = 0$
Total Volatility	62%	62%	15.3%	15%
Risk Premium	NA	NA	7%	6.8%
Jump Intensity $\lambda_1 = 1.5$				

	Exchange Rate e
Diffusion dZ_{1t}	$\sigma_{e1} = 0.07$
Diffusion dZ_{2t}	$\sigma_{e2} = 0.06$
Jump $d\mathcal{N}_{1t}$	$\Delta_{e1} = 0.04$
Total Volatility	10.6%
Carry Trade Premium	3.8%
$Corr_t\left(\frac{dM_H}{M_H}, \frac{dM_F}{M_F}\right)$	30%
$Corr_t\left(\frac{d\left(\frac{M_{Ht}}{M_{Ft}}\right)}{\frac{M_{Ht}}{M_{Ft}}}, \frac{de_t}{e_t}\right)$	14%

Notes: Given the quantities specifying the market prices of risk (risk loadings of SDFs M_H and M_F) and the risk exposures of the two traded assets Y_1 and Y_2, we determine the pricing-consistent exchange rate according to the system of two equations (30). In the case of economy (II) with entangled risks where multiple pricing-consistent exchange rates exist, we only report the one solution which best matches the first two moments of exchange rates in the data. $Corr_t\left(\frac{dM_H}{M_H}, \frac{dM_F}{M_F}\right) = \frac{Cov_t\left(\frac{dM_H}{M_H}, \frac{dM_F}{M_F}\right)}{Vol_t\left(\frac{dM_H}{M_H}\right) Vol_t\left(\frac{dM_F}{M_F}\right)}$ is the total correlation between the two SDFs, where $Cov_t\left(\frac{dM_H}{M_H}, \frac{dM_F}{M_F}\right) = \sum_{i=1}^{2} \eta_{Hi}\eta_{Fi} dt + \lambda_1 dt \left(e^{\Delta_{H1}} - 1\right)\left(e^{\Delta_{F1}} - 1\right)$ is the total covariance between the two SDFs and $Vol_t\left(\frac{dM_I}{M_I}\right) = \sqrt{\sum_{i=1}^{2} \eta_{Ii}^2 dt + \lambda_1 dt \left(e^{\Delta_{I1}} - 1\right)^2}$ is the total volatility of SDF I. The formulas to compute the total exchange rate volatility, carry trade premium and the correlation between the exchange rate growth and the growth in the ration of SDFs $\left(Corr_t\left(\frac{d\left(\frac{M_{Ht}}{M_{Ft}}\right)}{\frac{M_{Ht}}{M_{Ft}}}, \frac{de_t}{e_t}\right)\right)$ are given in (22), (23), (24).

(diffusion) volatility of Y_1 is about 15%, which roughly matches the unconditional volatility of the US stock market. We set the jump size $\Delta_{Y_1 1}$ of asset Y_1 equal to -2.5% and the jump intensity $\lambda_1 = 1.5$, which match the estimation by Backus et al. (2011). The total volatility increases only marginally after adding the jump. Moreover, we choose the risk loadings of SDFs such that the risk premium on Y_1 is 7% (in economy (II) with the jump and slightly less in economy (I) without the jump risk), which matches the average excess return of the US stock market. Y_2 is an additional generic asset, which helps to complete markets in economy (I). When risks are completely disentangled (economy (I)), then the specific risk loadings of Y_2 are irrelevant (as long as they are distinct from Y_1) because the unique pricing consistent exchange rate is fully determined by the home and foreign SDFs and is independent of the risk-asset configuration. In contrast, when risks are entangled (economy II) the specification of Y_2 matters. The foreign bond and thus the inverse of the exchange rate is a portfolio of the risk-free home bond and assets Y_1 and Y_2 according to equation (21), and thus, the specification of Y_2 is not innocuous to the determination of the pricing consistent exchange rates. Home and foreign risk-free rates do not affect the system of equations (30), and thus, we do not specify them in our analysis.

As mentioned above, once the SDFs and asset space are specified, we solve for the exchange rate according to the system of two equations (30). In economy (I) with completely disentangled risks we have a unique exchange rate. While the model is set up to produce a mild correlation between the two SDFs (30%, which is similar to correlations in consumption growths across developed countries), it produces an unreasonably large exchange rate volatility of 73.2% (Table 1). Not surprisingly this finding confirms the puzzle posed by Brandt et al. (2006). In addition, the carry trade premium is too large because the exchange rate is very volatile. The exchange rate is equal to the ratio of SDFs, and thus, the correlation between the exchange rate growth and the growth in the ratio of SDFs is 100%. Conditional on a basic view that SDF growths are monotonic in consumption growths,[24] this is contrary to Backus and Smith (1993)'s empirical finding that the ratio of consumption growths is disconnected from the exchange rate growth.

In contrast, in economy (II) with entangled risks we have multiple exchange rates which are consistent with no-arbitrage pricing. As aforementioned we only report one exchange rate solution in Table 2. That is we only report the solution which most closely matches the first two moments of exchange rates in the data. The solution reported in Table 2 addresses the puzzle posed by

[24] We abstract from sophisticated structural features for which SDFs are predominantly driven by (unobservable) state variables other than current consumption growth.

Brandt et al. (2006): the exchange rate is smooth (total volatility of 10.6%) and at the same time the total correlation between the two SDF growths is a modest 30%. Moreover, the carry trade premium is 3.8% which is much closer to the data than the premium in economy (I). The exchange rate is no longer equal to the ratio of SDFs and the correlation between the exchange rate growth and the growth in the ratio of SDFs is a moderate 14%, which reflects the disconnection between consumption growths and exchange rates documented by Backus and Smith (1993). We emphasize that none of the exchange rates in the jump-diffusion setting with entangled risks (neither the one reported in Table 2 nor the other less reasonable ones) coincides with the exchange rate obtained in the diffusion model with completely disentangle risks. That is, none of the solutions in economies (I) and (II) overlap.

Finally, we vary η_{H1} and η_{H2} to change the total correlation $Corr_t\left(\frac{dM_H}{M_H}, \frac{dM_F}{M_F}\right)$ between the two SDF growths.[25] Figure 1 plots the total volatility of the pricing consistent exchange rates in economies (I) and (II) against $Corr_t\left(\frac{dM_H}{M_H}, \frac{dM_F}{M_F}\right)$.

The solid red line plots the exchange rate volatility in economy (II) with entangled jump-diffusion risks, (again, we only plot the solution which we regard as the economically most reasonable one, i.e., with first and second moments closest to the data), while the dashed black line represents the exchange rate volatility in economy (I) with completely disentangled diffusion risks. The dashed black line illustrates the correlation puzzle: a mild correlation between the two SDF growths implies an unreasonably large variation in the exchange rate, or a reasonably smooth exchange rate comes with almost perfectly correlated SDFs. In contrast, the solid red line shows that independent of the total correlation between the two SDFs, the total volatility of the exchange rate is reasonably small in the case of entangled jump-diffusion risks. Therefore, an incomplete market setting with entangled risks is able to reconcile a smooth exchange rate and a low cross-country correlation in SDF growths. The solid red line in the top panel of Figure 2 shows that the expected carry trade return is around 5% when the correlation between the SDFs is low (around 0.25) in the economy with entangled risks (economy (II)). This value is similar to the average carry trade return of borrowing low and lending high interest rate currencies in the data. In contrast, the expected carry trade return is too large when the correlation between SDFs is moderate in the

[25] In particular, we fix the parameters $\eta_{F2} = 0.15$, $\sigma_{Y_1 1} = 0.109$, $\sigma_{Y_1 2} = 0.104$, $\sigma_{Y_2 1} = 0.11$, $\sigma_{Y_2 2} = 0.102$, $\lambda = 1.5$ and for economy (I) $\Delta_{H1} = \Delta_{F1} = \Delta_{Y_1 1} = \Delta_{Y_2 1} = 0$ and for economy (II) $\Delta_{H1} = \Delta_{F1} = 0.04$, $\Delta_{Y_1 1} = -0.025$, $\Delta_{Y_2 1} = 0$, while we vary $\eta_{H1} \in [-0.1, 0.5]$ and simultaneously adjust η_{H2} to keep the equity premium of the stock market Y_1 equal to 7% and set η_{F1} such that the variances of SDF growths of H and F are identical (i.e., $\eta_{F1}^2 + \eta_{F2}^2 = \eta_{H1}^2 + \eta_{H2}^2$).

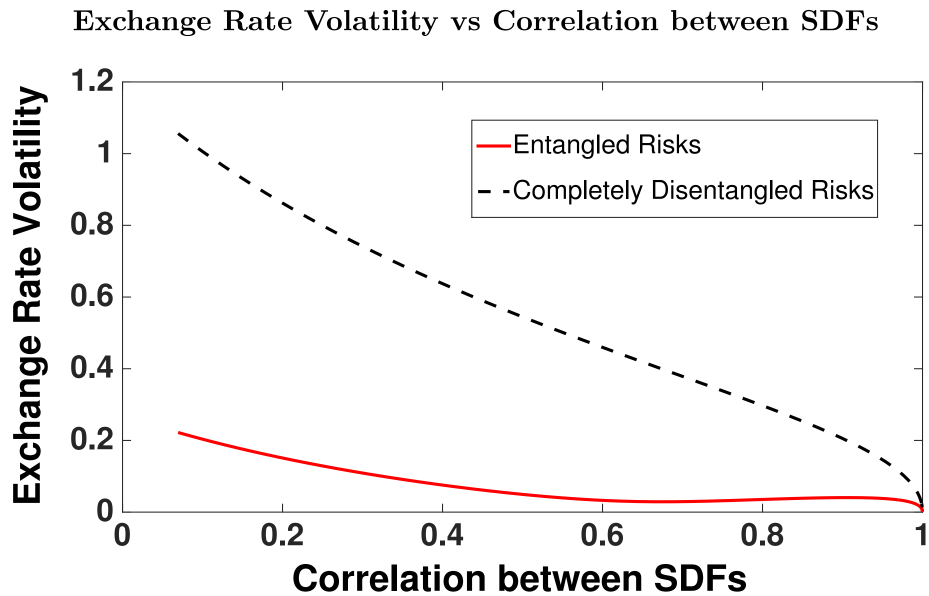

Figure 1: The dashed black line illustrates the difficulty to jointly have a smooth exchange rate growth (of 10% volatility) and mildly correlating SDF growths (of 30% correlation) in economy (I) with completely disentangled risks. The solid red line illustrates that the exchange rate can be smooth and SDF growths mildly correlated in economy (II) with entangled risks. In the case of economy (II) with entangled risks where multiple pricing-consistent exchange rates exist, we only report the one solution which best matches the first two moments of exchange rates in the data. We fix the parameters $\eta_{F2} = 0.15$, $\sigma_{Y_1 1} = 0.109$, $\sigma_{Y_1 2} = 0.104$, $\sigma_{Y_2 1} = 0.11$, $\sigma_{Y_2 2} = 0.102$, $\lambda = 1.5$ while we vary $\eta_{H1} \in [-0.1, 0.5]$ in order to vary $Corr_t\left(\frac{dM_H}{M_H}, \frac{dM_F}{M_F}\right)$. We simultaneously adjust η_{H2} to keep the equity premium of the stock market Y_1 equal to 7% and set η_{F1} such that variance of SDF growths of H and F are identical (i.e., $\eta_{F1}^2 + \eta_{F2}^2 = \eta_{H1}^2 + \eta_{H2}^2$). In the case of entangled risks (economy (II)) we set $\Delta_{H1} = \Delta_{F1} = 0.04$, $\Delta_{Y_1 1} = -0.025$, $\Delta_{Y_2 1} = 0$ and $\lambda = 1.5$. In the case of completely disentangled risks (economy (I) there are no jumps, i.e., all jump size parameters are zero, $\Delta_{H1} = \Delta_{F1} = \Delta_{Y_1 1} = \Delta_{Y_2 1} = 0$.

setting of completely disentangled risks (economy (I)) as illustrated by the dashed black line.

The bottom panel in Figure 2 further shows the correlation between changes in the exchange rate and changes in the ratio of country-specific SDFs. In the case of complete markets (economy (I), dashed black line), the exchange rate is equal to the ratio of SDFs and the correlation is 1. In the case of entangled risks (economy (II), solid red line), the exchange rate is distinct from the ratio of SDFs and the correlation between the two quantities is particularly low (between 0.1 and 0.2) when the correlation between the SDFs is low (less than 0.4).

We conclude that even in a simple setting of three risk sources and two non redundant assets, entangled jump-diffusion risks can generate several patterns which may be puzzling in a complete market setting. Namely, our example is able to reconcile the international correlation puzzle posed by Brandt et al. (2006), produce a reasonable carry trade premium and generate a disconnection

Carry Trade and Relationship between Exchange Rate and the Ratio of SDFs

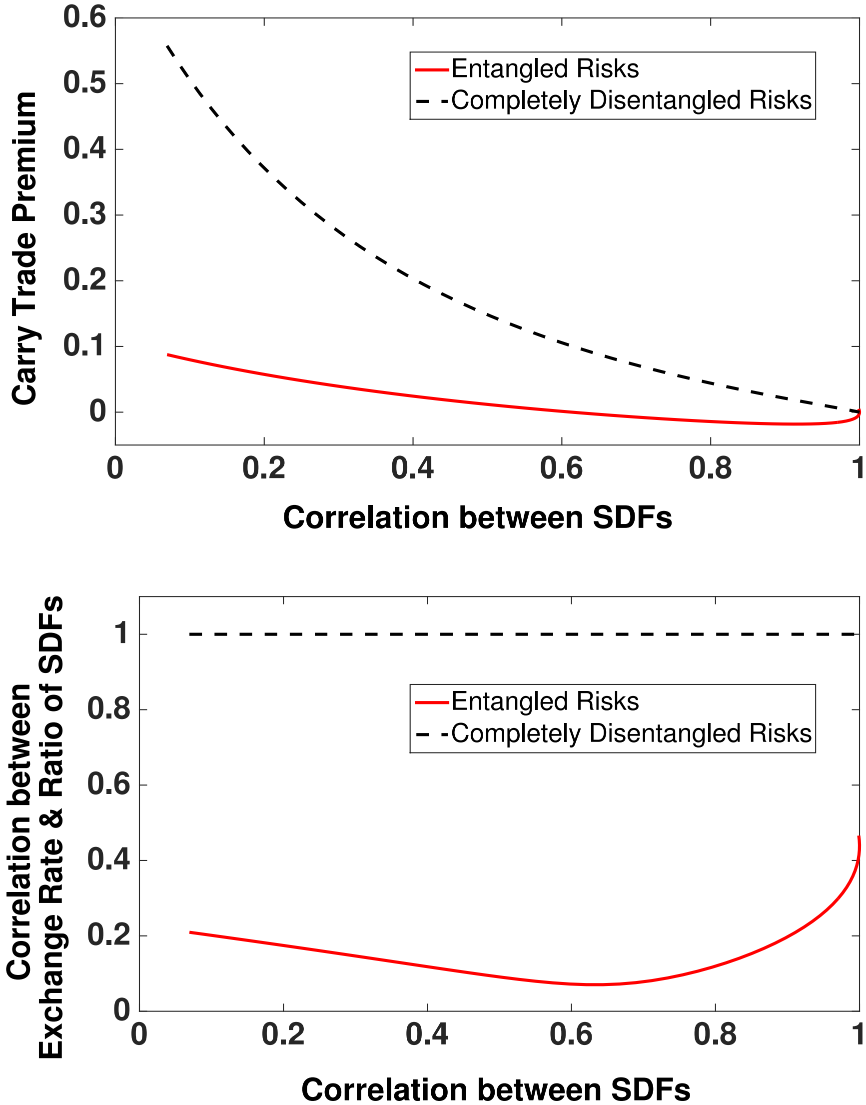

Figure 2: Top Panel: Expected carry trade return if risks are entangled (solid red line) or completely disentangled (dashed black line). Bottom Panel: Correlation between exchange rate growth and change in ratio of home and foreign SDFs if risks are entangled (solid red line) or completely disentangled (dashed black line). In the case of economy (II) with entangled risks where multiple pricing-consistent exchange rates exist, we only report the one solution which best matches the first two moments of exchange rates in the data. We fix the parameters $\eta_{F2} = 0.15$, $\sigma_{Y_1 1} = 0.109$, $\sigma_{Y_1 2} = 0.104$, $\sigma_{Y_2 1} = 0.11$, $\sigma_{Y_2 2} = 0.102$, $\lambda = 1.5$ while we vary $\eta_{H1} \in [-0.1, 0.5]$ in order to vary $Corr_t \left(\frac{dM_H}{M_H}, \frac{dM_F}{M_F} \right)$. We simultaneously adjust η_{H2} to keep the equity premium of the stock market Y_1 equal to 7% and set η_{F1} such that variance of SDF growths of H and F are identical (i.e., $\eta_{F1}^2 + \eta_{F2}^2 = \eta_{H1}^2 + \eta_{H2}^2$). In the case of entangled risks (economy (II)) we set $\Delta_{H1} = \Delta_{F1} = 0.04$, $\Delta_{Y_1 1} = -0.025$, $\Delta_{Y_2 1} = 0$ and $\lambda = 1.5$. In the case of completely disentangled risks (economy (I) there are no jumps, i.e., all jump size parameters are zero, $\Delta_{H1} = \Delta_{F1} = \Delta_{Y_1 1} = \Delta_{Y_2 1} = 0$.

between SDFs and the exchange rate in the spirit of Backus and Smith (1993).

4 Conclusion

We discuss the concept of risk entanglements in incomplete FX markets. We define risks as completely disentangled if there are sufficient traded assets to load singly on every risk in the space of conditional asset returns, and as entangled otherwise. Risk entanglement arises naturally and gives rise to surprising results. When risks are entangled in integrated international financial markets, there are multiple exchange rates that are consistent with given country-specific SDFs and asset returns. Furthermore, a higher degree of risk entanglement facilitates a larger number of pricing-consistent exchange rate solutions, and thus, higher flexibility to match the exchange rate in the data. The multiplicity of consistent exchange rates in the context of risk entanglement is not subject to the nuisance of ad-hoc selecting the desirable (data-matching) exchange rate solution ex-post. Instead, it guides us to the desirable asset market setting ex-ante, from which emerges uniquely the desirable exchange rate. We model a simple international asset market setting that features risk entanglement to calibrate three related regularities of international finance, namely the international correlation puzzle, the Backus-Smith puzzle, and large currency excess returns. Risk entanglement presents a novel market-based approach to model and rationalize international asset pricing.

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Appendices

A Further Details on Risk Entanglement: General Case

This appendix provides notational details and additional analysis for the risk entanglement in the general jump diffusion setting of Section 2.6.

On Assumption 1: The assumption on international market integration asserts that if a cash flow is originated and traded in a country, investors from all other countries can also trade this cash flow by converting it back and forth between the originated currency and their home currencies. Accordingly, a traded asset Y can be priced in any currency denomination,

$$E_t\left[\frac{M_{Ht+dt}}{M_{Ht}}\ \frac{Y_{t+dt}}{Y_t}\right] = E_t\left[\frac{M_{Ft+dt}}{M_{Ft}}\ \frac{e_{t+dt}}{e_t}\ \frac{Y_{t+dt}}{Y_t}\right] = 1, \tag{25}$$

where $\frac{Y_{t+dt}}{Y_t}$ denotes the asset's gross return in the home currency, and e_t the exchange rate in per-home-currency convention. Note that Assumption 1 does not preclude incomplete markets because e.g., traded asset returns might still not span individual innovations to SDFs M_{Ht}, M_{Ft}. This assumption on markets' perfect integration is natural, in particular, for developed economies.

The assumption on bonds' tradability models after another innocuous feature that investors in a country can participate in FX markets and, through them, in the short-term lending and borrowing of foreign currencies. In practice, it is this feature that underlies the viability and popularity of currency carry trades.

Jump-diffusion Specifications: To implement Protocol 1 in a generic jump-diffusion setting, SDF specifications are given as follows,

$$\frac{M_{It+dt}}{M_{It}} = 1 - r_{It}dt - \eta'_{It}dZ_t + \sum_{i\in\mathcal{J}_I}\left(e^{\Delta_{Iit}} - 1\right)\left(d\mathcal{N}_{it} - \lambda_{it}dt\right), \tag{26}$$

$$\text{with} \quad M_{I0} = 1, \quad t\in[0,\infty), \quad I\in\{H,F\}, \quad d\mathcal{N}_{it}\in \text{Poisson}(\lambda_{it}),$$

where d-dimensional standard Brownian motion Z_t captures d independent diffusion risks in our setting, and notation $'$ denotes the matrix transpose throughout. There are j different and uncorrelated types of jump risks indexed by $i\in\{1,\dots,j\}$,[26] and $\mathcal{J}_I$ denotes the set of jump risks

[26]Correlated jump types can be decomposed into and constructed from uncorrelated jump types.

priced by country I's investors (i.e., affecting M_I). A jump risk of particular type i is characterized by a Poisson counting process $\mathcal{N}_{it}$ of arrival intensity λ_{it}. Scalar Δ_{Iit} denotes the discontinuous change (i.e., jump size) in country I's SDF growth when a jump of type i occurs. Vector η_{It} of d dimensions denotes the prices of diffusion risks in respective country I. The expected growth of SDF M_I (26) is the additive inverse of the instantaneous risk-free rate $-r_{It}$ because M_I prices the risk-free bond of the respective country I. This SDF specification generalizes (1) to multiple risks.

Return specifications of $N+1$ basis assets $\{B_H, Y_n\}$, $n \in \{1, \dots, N\}$, in the home currency denomination are,

$$\frac{B_{Ht+dt}}{B_{Ht}} = 1 + r_H dt, \qquad \frac{Y_{t+dt}}{Y_t} = 1 + \mu_{Yt} dt + \sigma'_{Yt} dZ_t + \sum_{i \in \mathcal{J}_Y} \left(e^{\Delta_{Yit}} - 1\right)\left(d\mathcal{N}_{it} - \lambda_{it} dt\right), \tag{27}$$

where B_H denotes the home bond, $\mathcal{J}_Y$ the set of jump types that impact asset Y's return (in the home currency). Scalar μ_{Yt} and $d-$dimensional vector σ_{Yt} denote asset Y's expected return and return volatility respectively. Scalar Δ_{Yit} denotes the jump size in asset Y's return associated with a jump of type i.[27] Markets are possibly incomplete because jump-diffusion risks impacting either asset payoffs or SDFs are not perfectly hedged by holding portfolios of traded assets. These specifications generalize (4) to an arbitrary risk-asset embedding.

Portfolio Representation of the Exchange Rate: The representation (21) and asset return specifications (27) imply an exchange rate process,[28]

$$\frac{e_{t+dt}}{e_t} = 1 + \mu_e dt + \sigma'_e dZ_t + \sum_{i \in \mathcal{J}_{\{Y\}}} \left(e^{\Delta_{ei}} - 1\right) d\mathcal{N}_{it}, \tag{28}$$

with the exchange rate's drift, diffusion and jump components,

$$\mu_e = r_F - \left[1 - \sum_{Y \in \{Y\}} \alpha_Y\right] r_H + \sigma'_e \sigma_e - \sum_{Y \in \{Y\}} \alpha_Y \left[\mu_Y - \sum_{i \in \mathcal{J}_Y} \lambda_i \left(e^{\Delta_{Yi}} - 1\right)\right],$$

$$\sigma_e = - \sum_{Y \in \{Y\}} \alpha_Y \sigma_Y, \qquad e^{\Delta_{ei}} \equiv \frac{1}{1 + \sum_{Y \in \mathcal{Y}_i} \alpha_Y \left(e^{\Delta_{Yi}} - 1\right)}, \quad \forall i \in \mathcal{J}_{\{Y\}}. \tag{29}$$

[27] In general, $r_{It}, \eta_{It}, \Delta_{Iit}, \lambda_{it}, \mu_{Yt}, \sigma_{Yt}, \Delta_{Yit}$ are stochastic processes (adapted to the information structure generated by Z_t and $\{\mathcal{N}_{it}\}$, $\forall i$). Our analysis holds conditional on generic time t, henceforth, we will drop time indices whenever such an omission does not create ambiguities.

[28] Substituting (27) into (21) only yields an expression for the reciprocal of the exchange rate growth $\frac{e_t}{e_{t+dt}}$. We need to apply Itô's lemma to yield the multiplicative inverse of this ratio to obtain the proper exchange rate growth $\frac{e_{t+dt}}{e_t}$.

Above, $\mathcal{J}_{\{Y\}}$ the set of all jump types impacting asset returns $\{Y\}$, and $\mathcal{Y}_i$ the set of all assets affected by the jump of type i. Two observations are in order. First, (29) provides a concrete recipe to determine the exchange rate via the solution of weights $\{\alpha_Y\}$ (generalizing (7), (11) and (15)). Clearly, the exchange rate volatility σ_e found in (29) implies that diffusion risks impacting the exchange rate are identical to those impacting asset returns in the home currency. Furthermore, because the volatility of asset Y's return in the foreign currency equals $\sigma_e + \sigma_Y$, the diffusion risks impacting the asset returns in the foreign currency are also identical to those impacting the asset returns in the home currency. Hence, in the absence of jump risks (i.e., pure diffusion settings), asset return spaces denominated in home and foreign currencies are identical.

Remark 2 *The tradability of risk-free bonds (Assumption 1) implies that asset return spaces denominated in home and foreign currencies have identical diffusion subspaces.*

Second, any jump that affects returns on traded assets also enters the exchange rate dynamics in general as a result of (21). We also note that, exchange rate jumps do not appear in a compensated form in the convention of (28). It is its inverse, $(1+r_F dt)\frac{e_t}{e_{t+dt}}$ (which is a portfolio return, according to (21)), that has such a form.

The market construction of the exchange rate (Protocol 1) now becomes the determination of portfolio weights $\{\alpha_Y\}$ in the representation (21). We solve for $\{\alpha_Y\}$ by enforcing the pricing consistency equation (3) on every traded asset across the two currencies: (i) using exchange rate (28) to convert the asset returns specified in the home currency (27) to returns in the foreign currency, and (ii) combining foreign pricing equations (using M_F) on returns just constructed) with home pricing equations (using M_H). For a traded asset Y, this operation yields,

$$\begin{aligned}&\sigma_Y'\left(\eta_H - \eta_F + \sigma_e\right) + \textstyle\sum_{i\in(\mathcal{J}_Y\cap\mathcal{J}_F)} \lambda_i e^{\Delta_{Fi}+\Delta_{ei}}\left(e^{\Delta_{Yi}}-1\right) + \sum_{i\in(\mathcal{J}_Y\setminus\mathcal{J}_F)} \lambda_i e^{\Delta_{ei}}\left(e^{\Delta_{Yi}}-1\right)\\ &\quad = \textstyle\sum_{i\in(\mathcal{J}_Y\cap\mathcal{J}_H)} \lambda_i\left(e^{\Delta_{Hi}}-1\right)\left(e^{\Delta_{Yi}}-1\right) + \sum_{i\in\mathcal{J}_Y} \lambda_i\left(e^{\Delta_{Yi}}-1\right), \qquad \forall Y\in\{Y\}.\end{aligned} \tag{30}$$

First, given the asset return (i.e., σ_Y and Δ_{Yi}) and SDF specifications (i.e., η_H, η_F, Δ_{Hi} and Δ_{Fi}), (30) is an equation of portfolio weights $\{\alpha_Y\}$ (which determine exchange rate's diffusion and jump components σ_e, $e^{\Delta_{ei}}$ (29)). If we have N non-redundant risky assets, we have N non-redundant equations of type (30) (one equation per risky asset). Altogether, they form a system of N equations and N unknowns $\{\alpha_Y\}$ (with $Y\in\{Y\}$) that consistently determines the exchange rate (28) in our approach. This confirms the first part of Theorem 1.

Second, all jump types affecting SDFs M_{Ht} and M_{Ft} but not the asset return space $\{Y\}$ drop out from, and do not contribute to the determination of the exchange rate. This is because such jumps represent non-market risks, while Protocol 1 is a market-based approach by construction. All other risks in the asset return space $\{Y\}$, either of continuous (diffusion) or discontinuous (jump) nature, jointly determine the solution weights $\{\alpha_Y\}$ in the portfolio representation (21), and thus, the exchange rate.

Third, it is crucial to observe that, while $\sigma_e = -\sum_{Y\in\{Y\}} \alpha_Y \sigma_Y$ (29) is linear in portfolio weights $\{\alpha_Y\}$, the exchange rate's jump sizes $e^{\Delta_{ei}} = \frac{1}{1+\sum_{Y\in\mathcal{Y}_i} \alpha_Y \left(e^{\Delta_{Yi}}-1\right)}$ (29) are non-linear in $\{\alpha_Y\}$. Consequently, the set (30) for all traded assets also constitutes a system of N nonlinear equations. There potentially exist multiple solution sets $\{\alpha_Y\}$, and hence, potentially multiple pricing-consistent exchange rates of rich dynamics. The same non-linearity also gives rise to a quadratic equation and two exchange rate solutions in the incomplete market illustration of Section 2.4 (as discussed in Section 2.5).

We next discuss the key issue on the non-linearity of system (30), and hence, the multiplicity of exchange rates. As suggested by the discussion of Section 2.5 on specific illustrations of Section 2, the exchange rate multiplicity is crucially related to risk entanglement.

Complete Risk Disentanglement

Following Definition 1, two special cases of complete risk disentanglement are (i) complete markets and (ii) incomplete markets with pure-diffusion risks (see Remark 1). The general case of complete risk disentanglement can also be quantified. Generalizing the illustration (9) of Section 2.3, a primitive asset market configuration that completely disentangles d diffusion risks $Z_t' = (Z_{1t}, \ldots, Z_{\bar{d}t})$ and j types of (uncorrelated) jump risks $(d\mathcal{N}_{1t}, \ldots, d\mathcal{N}_{\bar{j}t})$ is,

$$\mathcal{T} = B_I \cup \{Y\}, \quad \text{with} \quad \{Y\} \equiv \{X_k, W_i\}, \;\; k \in \{1, \ldots, \bar{d}\};\; i \in \{1, \ldots, \bar{j}\}, \tag{31}$$

where primitive asset X_k loads only on k-th diffusion component dZ_{kt}, W_i only on i-th jump type $d\mathcal{N}_{it}$. This primitive market configuration $\mathcal{T}$ (31), by construction, completely disentangles risks in asset markets (Definition 1). However, such configuration is not unique or special in this role. In fact, infinitely many other configurations of (non-primitive) traded assets can also perform this task, and all of them are equivalent as the following result shows.

Proposition 1 *Every asset market configuration, in which these risks are completely disentangled, is isomorphic to the one spanned by the* $1 + \bar{d} + \bar{j}$ *primitive assets in set* $\mathcal{T}$ *(31).*

A proof is given in Appendix B. By virtue of this proposition, properties that can be proved using the primitive asset market configuration $\mathcal{T}$ remain valid for any other asset market configuration in which risks are completely disentangled. As an application of this result, it suffices to prove Theorem 1 by establishing its necessary and sufficient condition for the primitive asset market configuration $\mathcal{T}$ (see Appendix B).

In a market setting with completely disentangled risks, if asset Y's return in the home currency loads singly on the jump risk of type i, so does its return in the foreign currency. Therefore, the complete risk disentanglement is a characteristic that is invariant to currencies of denomination. The following remark states a version of this result.

Remark 3 *Assuming international market integration and bonds' tradability (Assumption 1), when risks are completely disentangled in asset markets (Definition 1), asset return spaces denominated in either currency are subject to identical risk-asset embedding configurations. As a results, under these premises, the two return spaces are identical.*

Intuitively, when there are enough assets to disentangle every individual risk affecting asset markets in a pair of countries, a respective pair of Euler equations (25) holds effectively for every individual priced risk in asset markets.[29] As a result, complete risk disentanglement completely decouples the equation system (30): an equation therein (say, pertaining asset Y) contains only one unknown weight (α_Y), yielding a single pricing-consistent exchange rate process (28) with the following components,

$$\text{Jump components:} \quad \begin{cases} \text{For } i \in \left(\mathcal{J}_{\{Y\}} \cap \mathcal{J}_H \cap \mathcal{J}_F\right) & : \quad \Delta_{ei} = \Delta_{Hi} - \Delta_{Fi}, \\ \text{For } i \in \left(\mathcal{J}_{\{Y\}} \cap \mathcal{J}_H\right) \setminus \mathcal{J}_F & : \quad \Delta_{ei} = \Delta_{Hi}, \\ \text{For } i \in \left(\mathcal{J}_{\{Y\}} \cap \mathcal{J}_F\right) \setminus \mathcal{J}_H & : \quad \Delta_{ei} = -\Delta_{Fi}, \\ \text{For } i \in \left(\mathcal{J}_{\{Y\}} \setminus \mathcal{J}_H\right) \setminus \mathcal{J}_F & : \quad \Delta_{ei} = 0. \end{cases} \tag{32}$$

$$\text{Diffusion and Drift components:} \qquad \sigma_{et} = \eta_{Ft\|} - \eta_{Ht\|},$$

[29] Originally, Euler pricing equations hold for traded assets. But when risks are completely disentangled in markets, each individual risk can be proxied by a traded asset (or a portfolio of traded assets), so that we can effectively associate the Euler equation (on the asset) with the individual risk that the asset loads on.

$$\mu_{et} = r_{Ft} - r_{Ht} + \eta'_{Ft\|}\sigma_{et} - \sum_{i\in(\mathcal{J}_{\{Y\}}\cap\mathcal{J}_H)} \lambda_i \left(e^{\Delta_{Hi}} - 1\right) - \sum_{i\in(\mathcal{J}_{\{Y\}}\cap\mathcal{J}_F)} \lambda_i \left(1 - e^{\Delta_{Fi}}\right), \qquad (33)$$

where $\eta_{I\|}$ denotes the vector of country I's prices of diffusion risks projected onto the space of asset return risks (denominated in respective currency I). Note that there are not necessarily enough assets to hedge all risks affecting SDFs, so that markets are still possibly incomplete. Complete risk disentanglement enforces the pricing consistency (across currencies) individually for every market risk impacting asset markets, resulting in the unique exchange rate determination at the individual risk level (but not at the asset level). Consequently, exchange rate dynamics (32)-(33) are functions of only SDFs' dynamics (but not of asset returns). This strong relationship between the exchange rate and the pricing dynamics in the presence of complete risk disentanglement can also be characterized succinctly by the identity $e_t = \frac{M_{H\|t}}{M_{F\|t}}$ (Section 2.3). This identity is a benchmark to imply apparent disparities between the dynamics of the observed exchange rate and the SDF ratio in international finance. The sufficient and necessary condition of Theorem 1 then implicates that a departure from complete risk disentanglement is the only pure market-based approach to neutralize this benchmark, and hence, rationalize the disconnection between the exchange rate and the pricing dynamics.

Risk Entanglement

The Definition 1 of risk entanglement formalizes the intuitions learned from the specific illustrations of Section 2 (and the discussion of Section 2.5). Risk entanglement goes beyond the standard notion of incomplete spanning in that it takes effect only in the presence of (some) risks of significant magnitudes, as the illustration and discussion underlying equation (17) indicate. Quantitatively, risks of significant magnitudes prescribe uncertain movements in asset prices for which the first (linear) order of the Taylor series is an inadequate approximation.[30] As a result, higher-order terms contribute and risk entanglement preserves the non-linearity nature of system (30), yielding multiple pricing-consistent exchange rates.[31] However, also note that risk entanglement does not require that every risk in asset returns has significant magnitudes, though the depth of risk entanglement varies with the amount of significant risks in markets.

As we observed earlier (Section 2.5 and the discussion following key equation (30)), the multi-

[30] That is when the approximation (which also underlies the illustration (17)), $\left(1 + \frac{dY_{t+dt}}{Y_t}\right)^{-1} \approx 1 - \frac{dY_{t+dt}}{Y_t}$, breaks down.

[31] Alternatively, when magnitudes of risks are insignificant (i.e., pure diffusion risks in Remark 1, or jump risks of small jump sizes in approximation (17)), risk entanglement does not arise even when markets are incomplete.

plicity of pricing-consistent exchange rates in the risk entanglement approach does not amount to multiple equilibria of the international economy. Instead, each exchange rate solution e identifies a distinct ex-post specification of the foreign bond return in the home currency, $(1 + r_F dt)\frac{e_t}{e_{t+dt}}$ (21), in terms of basis assets. Ex-ante, if we start out with a foreign bond specification associated with an particular exchange rate solution e_t, then the unique non-trivial exchange rate consistent with the given inputs $\{M_H, M_F, \{Y\}\}$ of Protocol 1 is the particular choice e_t we started out with (as illustrated below (19)-(20)). We elaborate on further aspects of Theorem 1 to relate it with specific illustrations of Section 2.

First, idiosyncratic risks (which impact asset payoffs but not the SDF of either country) may impact all solutions of pricing-consistent exchange rates. This is a surprising result, given that these idiosyncratic risks are not priced in expected asset returns in either currency, and the exchange rate is constructed from these returns. The reason is that idiosyncratic risks are entangled with systematic risks (which impact asset payoffs as well as SDFs) when markets are sufficiently incomplete and fall into the classification of Definition 1. As a result, there are not enough assets to disentangle each (idiosyncratic as well as systemic) risk affecting asset markets (though idiosyncratic risks are not priced in any country). Therefore, both systematic and idiosyncratic risks enter the constructed exchange rates via their entanglement. Such a scenario is intuitive and plausible because investors do not have incentives to extend markets to trade idiosyncratic risks, which do not affect their marginal utilities.

Second, system (30) determining the exchange rate is a balanced system of same numbers of polynomial equations and of unknown weights $\{\alpha_Y\}$, which incidentally also is the number N of traded risky assets (from the home perspective). The level of risk entanglement in asset markets, however, is commensurate with the imbalance $d + j - N > 0$ between the number of risk dimensions and available traded assets: a larger imbalance indicates a higher level of risk entanglement. Incidentally, a larger imbalance also marks a higher degree of polynomial equations in the key system (30), yielding a larger number of exchange rate solutions and thus a higher flexibility to obtain an exchange rate that matches the data.

Finally, in contrast to completely disentangled risks (Remark 3), entangled risks give rise to distinct asset return spaces when returns are denominated in different currencies. When risks are entangled in asset markets, the exchange rate's jump size associated with some type i is necessarily a (irreducible) combination of jump sizes in multiple assets (see (29)). As a result, at least some jump components of asset returns denominated in the foreign currency (i.e., returns on $\{eY\}$) are some (different) combinations of jump sizes in multiple assets. Because there are not enough assets loading singly on each jump type and at least some jump sizes are significant (i.e., entanglement), these two sets of combinations are not equivalent.

Remark 4 *Assuming international market integration and bonds' tradability (Assumption 1), when risks are entangled in asset markets (Definition 1), asset return spaces denominated in different currencies are subject to different risk-asset embedding configurations. As a results, under these premises, the two return spaces are distinct.*

Equivalently, under market integration, when risk entanglement holds for asset returns denominated in one currency, it holds for asset returns in all currencies. The existence of risk entanglement hence is invariant to the currency denomination, though the specific risk entanglement configuration varies with the denomination currency. This deviation between asset return spaces in different denomination currencies gives rise to the deviation between exchange rate and pricing dynamics, and promotes risk entanglement as a market venue to reconcile several international finance regularities.

B Proofs and Derivations

Equation System Determining Consistent Exchange Rates

The key equation system (30) is established by combining Euler pricing equations (in both currency denominations) of bond and risky assets. It is instructive to account for all pricing equations leading to this system.

The pricing of the short-term bond B_I in the respective currency I holds by specifying M_I's expected growth rate to be the additive inverse of the risk free rate (26), for $I \in \{H, F\}$. Similarly, the pricing of the foreign bond in the home currency, $E_t\left[\frac{M_{H,t+dt}}{M_{H,t}}\frac{B_{F,t+dt}}{B_{F,t}}\frac{e_t}{e_{t+dt}}\right] = 1$, is automatically enforced by the representation (21), in which the foreign bond's gross return to home investors, $\frac{B_{F,t+dt}}{B_{F,t}}\frac{e_t}{e_{t+dt}}$, is a proper portfolio return and hence is priced by M_H.

The pricing of the home bond B_H in the foreign currency, $E_t\left[\frac{M_{F,t+dt}}{M_{F,t}}\frac{e_{t+dt}}{e_t}(1 + r_H dt)\right] = 1$, can be rewritten as the premium to foreign investors on the home bond,

$$(\mu_e + r_H) - r_F = \sigma_e' \eta_F \quad - \sum_{i \in \left(\mathcal{J}_{\{Y\}} \cap \mathcal{J}_F\right)} \lambda_i \left(e^{\Delta_{Fi}} - 1\right)\left(e^{\Delta_{ei}} - 1\right) - \sum_{i \in \mathcal{J}_{\{Y\}}} \lambda_i \left(e^{\Delta_{ei}} - 1\right), \qquad (34)$$

where $\mathcal{J}_{\{Y\}}$ defined below (29) is the set of all jump types in the asset return space, and $\mathcal{J}_{\{Y\}} \cap \mathcal{J}_I$ set of jump types that affect both asset Y's return and country I's SDF.[32]

The pricing of risky asset Y (27) in the home currency, $1 = E_t\left[\frac{M_{H,t+dt}}{M_{H,t}} \frac{Y_{t+dt}}{Y_t}\right]$, implies the premium on this asset to home investors for taking diffusion and jump risks,

$$\mu_Y - r_H = \sigma_Y' \eta_H \quad - \sum_{i \in (\mathcal{J}_Y \cap \mathcal{J}_H)} \lambda_i \left(e^{\Delta_{Hi}} - 1\right)\left(e^{\Delta_{Yi}} - 1\right), \qquad (35)$$

where $\mathcal{J}_Y \cap \mathcal{J}_H$ is the set of jump types affecting both Y's return and the home SDF as in (34).

The pricing of the same risky asset Y in the foreign currency, $1 = E_t\left[\frac{M_{F,t+dt}}{M_{F,t}} \frac{e_{t+dt}}{e_t} \frac{Y_{t+dt}}{Y_t}\right]$, yields the premium on this asset to foreign investors,

$$\begin{aligned}(\mu_e + \mu_Y) - r_F = (\sigma_Y' + \sigma_e')\,\eta_F - \sigma_Y' \sigma_e - \textstyle\sum_{i \in (\mathcal{J}_Y \cap \mathcal{J}_F)} \lambda_i \left(e^{\Delta_{Fi} + \Delta_{Yi} + \Delta_{ei}} - 1\right) \\ - \textstyle\sum_{i \in \left(\mathcal{J}_{\{Y\}} \cap \mathcal{J}_F \setminus \mathcal{J}_Y\right)} \lambda_i \left(e^{\Delta_{Fi} + \Delta_{ei}} - 1\right) - \textstyle\sum_{i \in (\mathcal{J}_Y \setminus \mathcal{J}_F)} \lambda_i \left(e^{\Delta_{Yi} + \Delta_{ei}} - 1\right) \qquad (36) \\ + \textstyle\sum_{i \in \left(\mathcal{J}_{\{Y\}} \cap \mathcal{J}_F\right)} \lambda_i \left(e^{\Delta_{Fi}} - 1\right) + \textstyle\sum_{i \in \mathcal{J}_Y} \lambda_i \left(e^{\Delta_{Yi}} - 1\right) - \textstyle\sum_{i \in \left(\mathcal{J}_{\{Y\}} \setminus \mathcal{J}_Y \setminus \mathcal{J}_F\right)} \lambda_i \left(e^{\Delta_{ei}} - 1\right),\end{aligned}$$

where μ_e and σ_e denote the exchange rate's drift and diffusion components (29), and $(\mathcal{J}_Y \setminus \mathcal{J}_F)$ the set of jumps affecting Y's return but not the foreign SDF M_F. Similarly, $\left(\mathcal{J}_{\{Y\}} \cap \mathcal{J}_F \setminus \mathcal{J}_Y\right)$ denotes the set of jumps affecting the asset return space and the foreign SDF M_F but not the return on the particular asset Y under consideration, and $\left(\mathcal{J}_{\{Y\}} \setminus \mathcal{J}_Y \setminus \mathcal{J}_F\right)$ the set of jumps affecting the asset return space but neither asset Y nor M_F.[33]

[32] The left-hand side of (34) is the excess return (premium) on the home bond to foreign investors, who earn the bond's intrinsic interest r_H on top of an expected growth μ_e of the exchange rate. This premium is intuitive. Foreign investors holding the home bond are exposed exclusively to exchange rate movements and compensated for loading on (i) diffusion risks (when the diffusion moves the foreign currency's value and the foreign SDF in the same direction, $\sigma_e' \eta_F > 0$), and (ii) jump risks (when common jumps move the foreign currency's value and the foreign SDF in the same direction, $\left(e^{\Delta_{Fi}} - 1\right)\left(e^{\Delta_{ei}} - 1\right) < 0$).

[33] This premium is intuitive. On the right-hand side of (36), the term associated with η_F is the compensation on (fundamental and exchange rate) diffusion risks $(\sigma_Y + \sigma_e)$. Asset Y's hedging benefit towards exchange rate's diffusion risks (i.e., when $\sigma_Y' \sigma_e > 0$) reassuringly lowers Y's premium to foreign investors. When jumps common to all SDF M_{Ft}, the asset payoff Y_t, and the exchange rate e_t are such that $\Delta_{Fi} + \Delta_{Yi} + \Delta_{ei} > 0$, Y is a net hedge

To arrive at equation system (30), first decompose asset Y's premium to foreign investors into (i) Y's premium to home investors and (ii) the home bond's premium to foreign investors,

$$\underbrace{(\mu_e + \mu_Y) - r_F}_{\text{given in (36)}} \quad = \quad \underbrace{[\mu_Y - r_H]}_{\text{given in (35)}} \quad + \quad \underbrace{[(\mu_e + r_H) - r_F]}_{\text{given in (34)}}. \tag{37}$$

The substitution of premia (36), (35), (34) obtained earlier yields (30).

Finally, because the exchange rate is to be determined from the system (30), it is an important consistency check that its solution $\{\alpha_Y\}$ satisfies separately each of original pricing equations (34) and (36).[34] The following result confirms this consistency.

Proposition 2 *Assume that a set of weights $\{\alpha_Y\}$, and the associated exchange rate e_t constructed from these weights via equations (28)-(29), are a solution to the system of equations (30) on all traded assets $Y \in \{Y\}$. Then for such $\{\alpha_Y\}$ and e_t, each of pricing equations (34) and (36) separately holds.*

Proof 1 *It suffices to show that any solution to the system (30) is also a solution to the foreign pricing of the home bond (34). The other pricing equation (36) follows immediately from identity (37). Suppose $\{\alpha_Y\}$, and the associated exchange rate e_t (28)-(29), solve system (30). Because (30) is the explicit expression of the no-arbitrage pricing relationship (37), $\{\alpha_Y\}$ and e_t must also satisfy the latter, and its equivalent version (in terms of Euler equations),*

$$\underbrace{E_t\left[\frac{M_{Ft+dt}}{M_{Ft}}\frac{e_{t+dt}}{e_t}\frac{Y_{t+dt}}{Y_t}\right] - 1}_{\textit{Euler pricing eq. (36)}} \quad = \quad \underbrace{E_t\left[\frac{M_{Ht+dt}}{M_{Ht}}\frac{Y_{t+dt}}{Y_t}\right] - 1}_{\textit{Euler pricing eq. (35)}} \quad + \quad \underbrace{E_t\left[\frac{M_{Ft+dt}}{M_{Ft}}\frac{e_{t+dt}}{e_t}\frac{B_{Ht+dt}}{B_{Ht}}\right] - 1}_{\textit{Euler pricing eq. (34)}}.$$

By rearranging terms, the above equation can be rewritten as

$$E_t\left[\frac{M_{Ft+dt}}{M_{Ft}}\frac{e_{t+dt}}{e_t}\left\{\frac{Y_{t+dt}}{Y_t} - \frac{B_{Ht+dt}}{B_{Ht}}\right\}\right] \quad = \quad E_t\left[\frac{M_{Ht+dt}}{M_{Ht}}\frac{Y_{t+dt}}{Y_t}\right] - 1.$$

Note that the right-hand side is identically zero (implied from the Euler equation on the traded asset Y) – a property that has nothing to do with the solution of system (30). Consequently, multiplying

asset to foreign investors against these jump risks, yielding a lower premium, $-\left(e^{\Delta_{Fi}+\Delta_{Yi}+\Delta_{ei}} - 1\right) < 0$.

[34] If at least one of these two pricing equations does not hold, the exchange rate e_t (28) constructed from portfolio weights $\{\alpha_Y\}$ fails to price the home bond or some other traded asset $Y \in \{Y\}$ correctly (even though this exchange rate e_t is able to deliver the compounded pricing equation (30)), i.e., inconsistent-pricing solution e_t.

both sides by weight α_Y, then summing over all Y in the traded risky asset space $\{Y\}$ yields,

$$E_t\left[\frac{M_{Ft+dt}}{M_{Ft}}\frac{e_{t+dt}}{e_t}\left\{\left(\sum_{Y\in\{Y\}}\alpha_Y\frac{Y_{t+dt}}{Y_t}\right) - \frac{B_{Ht+dt}}{B_{Ht}}\sum_{Y\in\{Y\}}\alpha_Y\right\}\right]=0,$$

or equivalently,

$$E_t\left[\frac{M_{Ft+dt}}{M_{Ft}}\frac{e_{t+dt}}{e_t}\left\{\left(\sum_{Y\in\{Y\}}\alpha_Y\frac{Y_{t+dt}}{Y_t}\right) + \frac{B_{Ht+dt}}{B_{Ht}}\left(1-\sum_{Y\in\{Y\}}\alpha_Y\right)\right\}\right]=E_t\left[\frac{M_{Ft+dt}}{M_{Ft}}\frac{e_{t+dt}}{e_t}\frac{B_{Ht+dt}}{B_{Ht}}\right].$$

The portfolio representation (21) of the exchange rate, which underlies the no-arbitrage determination of the exchange rate (Protocol 1), then transforms above equation into,

$$E_t\left[\frac{M_{Ft+dt}}{M_{Ft}}\frac{e_{t+dt}}{e_t}\left\{\frac{B_{F,t+dt}}{B_{F,t}}\frac{e_t}{e_{t+dt}}\right\}\right]=E_t\left[\frac{M_{Ft+dt}}{M_{Ft}}\frac{e_{t+dt}}{e_t}\frac{B_{Ht+dt}}{B_{Ht}}\right].$$

After the cancellation of the exchange rate, the left-hand side of above equation is identically one (as an Euler equation associated with the foreign pricing of the foreign bond) – a property that has nothing to do with the solution of system (30). Hence, the above equation reduces to $1 = E_t\left[\frac{M_{Ft+dt}}{M_{Ft}}\frac{e_{t+dt}}{e_t}\frac{B_{Ht+dt}}{B_{Ht}}\right]$, which yields (34). This consistency result establishes the rigor of the system (30) in the determination of the exchange rate

Proof of Proposition 1: Without loss of generality, the proof is from the perspective of investors in country I. By definition, each of $d+j$ primitive risky assets in set $\mathcal{T}$ (31) loads only on one of $d+j$ jump-diffusion risks in asset return space. Hence, returns on primitive risky assets read,

$$\begin{aligned}\tfrac{X_{kt+dt}}{X_{kt}} &= 1+\mu_k dt+\sigma_k dZ_{kt}, \qquad \mu_k = r_I+\sigma_k\eta_{Ik},\quad k\in\{1,\ldots,d\},\\ \tfrac{W_{it+dt}}{W_{it}} &= 1+\mu_i dt+\left(e^{\Delta_{Wi}}-1\right)\left(d\mathcal{N}_{it}-\lambda_i dt\right),\\ \mu_i &= r_I-\lambda_i\left(e^{\Delta_{Ii}}-1\right)\left(e^{\Delta_{Wi}}-1\right),\quad i\in\{1,\ldots,j\}.\end{aligned} \tag{38}$$

By construction, obviously primitive assets (and the bond) in set $\mathcal{T}$ completely disentangle (i.e., singly replicate) any single risk in the set of d diffusion risks $\{dZ_{kt}\}_{k=1}^d$ and j types of jump risks $\{d\mathcal{N}_{it}\}_{i=1}^j$. To prove Proposition 1 we then just need to show that primitive assets in $\mathcal{T}$ can span any arbitrary asset return that bears these (and only these) $d+j$ risks in any possible way. This is because the set $\mathcal{A}$ of all these arbitrary assets is the most complete possible set as long as the $d+j$ asset market risks are concerned, and thus, these risks must be completely disentangled in the set

$\mathcal{A}$.[35]

Let's consider an arbitrary gross realized return $\frac{A_{t+dt}}{A_t}$ from $\mathcal{A}$,

$$\frac{A_{t+dt}}{A_t} = 1 + \mu_A dt + \sigma'_A dZ_t + \sum_{i=1}^{j} \left(e^{\Delta_{Ai}} - 1\right)\left(d\mathcal{N}_{it} - \lambda_i dt\right).$$

We now explicitly construct a portfolio P of weights $\{\beta_B, \beta_k, \beta_i\}_{k=1,i=1}^{d,j}$, respectively associated with primitive assets $\{B_I, X_k, W_i\}_{k=1,i=1}^{d,j}$ in $\mathcal{T}$ (31)-(38),

$$\frac{P_{t+dt}}{P_t} = 1 + \beta_B r_I dt + \sum_{k=1}^{d} \beta_k \left[\mu_k dt + \sigma_k dZ_{kt}\right] + \sum_{k=1}^{j} \beta_i \left[\mu_i dt + \left(e^{\Delta_{Wi}} - 1\right)\left(d\mathcal{N}_{it} - \lambda_i dt\right)\right],$$

$$\text{with portfolio normalization:} \qquad \beta_B + \sum_{k=1}^{d} \beta_k + \sum_{k=1}^{j} \beta_i = 1, \tag{39}$$

that perfectly replicates the arbitrary return $\frac{A_{t+dt}}{A_t}$. In order, we match diffusion, jump, and free (drift) components of $\frac{A_{t+dt}}{A_t}$ and $\frac{P_{t+dt}}{P_t}$.

Matching diffusion components: Because primitive asset X_k loads on a single diffusion component dZ_{kt} (38), the respective weight β_k in the replicating portfolio P is immediate and unique,

$$\sigma_{Ak} = \beta_k \sigma_k \implies \beta_k = \frac{\sigma_{Ak}}{\sigma_k}, \qquad \forall k \in \{1, \ldots, d\}.$$

Matching jump components: Similarly, because primitive asset W_i loads on a single type of jump $d\mathcal{N}_{it}$ (38), the matching equation is simple. Crucially, we note that because jumps of two (or more) different types almost surely do not jump together within an infinitesimal time span of dt. Therefore, we need to match the changes in returns $\frac{A_{t+dt}}{A_t}$ and $\frac{P_{t+dt}}{P_t}$ induced by each (and every) jump type i separately.[36] When a jump takes place, respective counter $d\mathcal{N}_{it}$ increases from 0 to 1 (while all other counters $\{d\mathcal{N}_{lt}\}_{l\neq i}$ remain at 0), so the matching of jump-induced changes in returns implies the respective weight β_i in the replicating portfolio P,

$$\left(e^{\Delta_{Ai}} - 1\right) = \beta_i \left(e^{\Delta_{Wi}} - 1\right) \implies \beta_i = \frac{e^{\Delta_{Ai}} - 1}{e^{\Delta_{Wi}} - 1}, \qquad \forall i \in \{1, \ldots, j\}.$$

It is important to observe that, by forming a portfolio (e.g., of a risk-free bond with an asset W_i

[35] Though markets are still possibly incomplete because some risks affecting the SDFs are not in $\mathcal{A}$.

[36] The reason we care primarily about the changes (of two returns to be matched) induced by jumps is that the no-jump (base) levels are accounted for in, and including in the matching of, the free components. See next.

sensitive to jump type i), one can replicate and transform the original asset's jump size Δ_{Wi} to an arbitrary jump size Δ_{Ai} associated with the same jump type i.

Matching free components: the weight associated with the risk-free bond is implied from weights $\{\beta_k, \beta_i\}$ found earlier via the normalization (39). Then, by virtue of no-arbitrage, the free terms (no-jump terms associated with dt while setting all jump counters $\{d\mathcal{N}_{it}\}$ to zero) are automatically matched,

$$\mu_A - \sum_{i=1}^{j} \lambda_i \left(e^{\Delta_{Ai}} - 1\right) = \beta_B r_I + \sum_{k=1}^{d} \beta_k \mu_k + \sum_{i=1}^{j} \beta_i \left[\mu_i - \lambda_i \left(e^{\Delta_{Wi}} - 1\right)\right],$$

This is because once the risk terms of two traded portfolios are matched, their expected returns (i.e., free terms) must also match by no arbitrage ■[37]

Proof of Theorem 1: The key equation system determining the exchange rate is (30). We take asset markets with completely disentangled risks in the specific configuration $\mathcal{T}$ (31) without loss of generality. To isolate d diffusion risks, we apply equation (30) on each of the d pure-diffusion assets X_k ($k \in \{1, \ldots, d\}$). Because these assets do not load on jump risks ($\Delta_{X_k i} = 0$, $\forall k, i$), (30) reduces to a $d-$dimensional identity,

$$\eta_{H\|} - \eta_{F\|} + \sigma_e = 0,$$

where $v_\|$ denotes the projected components of a generic vector v onto the asset return space. The above equation delivers the familiar relationship between volatilities of SDFs and the exchange rate in a pure-diffusion setting (see also Appendix D). Combining this identity with the foreign pricing of the home bond (34) yields the exchange rate's expected growth μ_e in (33).

To isolate the j types of jump risks, we surgically apply equation (30) on each (and every) asset W_i that loads only on a single jump type i that affects both home and foreign SDFs ($i \in \mathcal{J}_Y \cap \mathcal{J}_H \cap \mathcal{J}_F$). Such an asset retains only the jump terms associated with type i in (30), or,

$$\lambda_i e^{\Delta_{Fi} + \Delta_{ei}} \left(e^{\Delta_{Wi}} - 1\right) = \lambda_i \left(e^{\Delta_{Hi}} - 1\right)\left(e^{\Delta_{Wi}} - 1\right) + \lambda_i \left(e^{\Delta_{Wi}} - 1\right), \qquad i \in \mathcal{J}_Y \cap \mathcal{J}_H \cap \mathcal{J}_F.$$

Canceling common factor $\lambda_i \left(e^{\Delta_{Wi}} - 1\right)$ from both sides yields $e^{\Delta_{Fi} + \Delta_{ei}} = e^{\Delta_{Hi}}$, or equivalently

[37] We can also directly verify the matching of the free terms using the returns (38), the expression for the bond weight β_B, and the fact that as arbitrary asset A is traded, its expected return μ_A must satisfy the generic relationship (35) applied to A.

the first identity in (32).

Similarly, surgically applying equation (30) on each (and every) asset W_i that loads only on a single jump type i that affects:

a. the home but not the foreign SDF ($i \in \mathcal{J}_Y \cap \mathcal{J}_H \setminus \mathcal{J}_F$) yields the second identity in (32),

b. the foreign but not the home SDF ($i \in \mathcal{J}_Y \cap \mathcal{J}_F \setminus \mathcal{J}_H$) yields the third identity in (32),

c. neither the home nor the foreign SDF ($i \in \mathcal{J}_Y \setminus \mathcal{J}_H \setminus \mathcal{J}_F$) yields the last identity in (32).

Substituting this exchange rate jump size configuration into the home bond premium (34) in the foreign currency implies the exchange rate drift μ_e (33). The uniqueness of the exchange rate's components ($\{\Delta_{ei}\}, \sigma_e, \mu_e$) satisfying equations (32)-(33) is self-evident. Proposition 2, then, qualifies this uniquely constructed e_t as a pricing-consistent exchange rate ■

C Risk Entanglement in Discrete Settings

Risk entanglement arises not only in continuous but also in discrete settings of states and time. This appendix illustrates risk entanglement in discrete settings.

Set-up: From the perspective of current period t, we consider a discrete risk space $\mathcal{S} = \{1, \dots, S\}$ of S possible states at the next period $t+1$. The associated (conditional) probability distribution of future states is $\{p_t(s)\}$. Consider two countries (home and foreign) $I \in \{H, F\}$. We assume no-arbitrage opportunities, market integration and tradability of all countries' bonds. The countries' SDFs are given and specified as follows,

$$\widehat{m}_{Ht+1}(s) = \frac{M_{Ht+1}(s)}{M_{Ht}(s)}, \qquad \widehat{m}_{Ft+1}(s) = \frac{M_{Ft+1}(s)}{M_{Ft}(s)}, \quad \forall s \in \mathcal{S}. \tag{40}$$

While the international financial markets are possibly incomplete, they are perfectly integrated so that investors in either country trade the same set of financial assets, which include country-specific risk-free bonds. For concreteness and without loss of generality, we model the financial market as a set of $N+1$ non-redundant basis assets from home investors' perspective. Asset gross returns specified in the home currency denomination are given,

$$\frac{B_{Ht+1}(s)}{B_{Ht}} = 1 + r_{Ht}, \qquad \frac{Y_{nt+1}(s)}{Y_{nt}} = \widehat{y}_{nt+1}(s), \qquad \forall s \in \mathcal{S}, \quad \forall n \in \{1, \dots, N\}. \tag{41}$$

Above, B_H denotes the home risk-free bond, and $\{Y_n\}$ ($n \in \{1, \ldots, N\}$) denote N home risky assets. These $N+1$ basis assets span all traded assets. Markets are possibly incomplete because $N+1 \leq S$. Specifications (40), (41) and assumptions of this set-up are modeled after Protocol 1.

Pricing-Consistent Exchange Rates: Let the exchange rate e_t be the amount of the foreign currency F that buys one unit of the home currency H at time t. Since the foreign bond B_F is a traded asset in perfectly integrated international markets, its return in the home currency is spanned by basis returns. This bond tradability condition turns into a portfolio representation for the exchange rate. For self-consistence of this appendix, below we reproduce the representation (21) for discrete settings,

$$\frac{e_t}{e_{t+1}(s)} \frac{B_{Ft+1}}{B_{Ft}} = \left(1 - \sum_{n=1}^{N} \alpha_n\right) \frac{B_{Ht+1}}{B_{Ht}} + \sum_{n=1}^{N} \alpha_n \frac{Y_{nt+1}(s)}{Y_{nt}}, \qquad \forall s \in \mathcal{S}. \tag{42}$$

Consequently, the exchange rate determination is translated into solving for weights $\{\alpha_n\}$. The latter can be achieved through the requirement that every traded asset be priced consistently across currency denominations. Again, under the market integration assumption, foreign investors can trade, and hence M_F prices, all N risky returns in the foreign currency. That is, $E_t\left[\frac{M_{Ft+1}}{M_{Ft}}\left(\frac{e_{t+1}}{e_t}\frac{Y_{nt+1}}{Y_{nt}}\right)\right] = 1$, $\forall n \in \{1, \ldots, N\}$. Employing the "hat" notation for growth quantities introduced above, the representation (42) turns these pricing equations explicitly into,

$$\sum_{s \in \mathcal{S}} \frac{p_t(s)\, \widehat{m}_{Ft+1}(s)\, \widehat{y}_{nt+1}(s)}{1 + \sum_{k=1}^{N} \alpha_k \left[\frac{\widehat{y}_{kt+1}(s)}{1+r_{Ht}} - 1\right]} = \frac{1 + r_{Ht}}{1 + r_{Ft}}, \quad \forall n \in \{1, \ldots, N\}. \tag{43}$$

Given SDF growths $\widehat{m}_H$, $\widehat{m}_F$, and asset returns $\{\widehat{y}_n\}$ in the home currency, (43) is a system of N equations and N unknown weights $\{\alpha_n\}$ to determine the exchange rate (42). Similar to the equation system (30) of continuous settings, equations in (43) generally are non-linear in unknowns $\{\alpha_n\}$, and therefore, there exist possibly multiple exchange rate solutions. Risk entanglement conceptually characterizes this non-linearity, and therefore is the key to a rich set of pricing-consistent exchange rates. Formally, risk entanglement is the necessary and sufficient condition for system (43) to be genuinely non-linear (and thus, having multiple exchange rate solutions). The alternative premise, i.e., complete risk disentanglement, characterizes the conditions under which (43) is reduced to a linear system, yielding a unique solution for the exchange rate. We examine these two asset market premises in turn.

Completely Disentangled Risks: We observe that in discrete time and state settings, there are two scenarios, in which system (43) is linear in unknowns $\{\alpha_n\}$.

First scenario: In the first scenario, all S states of the asset risk space are completely spanned by $N+1$ traded assets $\{B_H, Y_n\}$. However, SDFs M_H, M_F can differentiate all S states, and thus investors are impacted differently by these S risks. In other words, investors are not able to hedge all relevant risks to them by trading assets, i.e., markets are possibly incomplete ($S \geq N+1$).

In this situation, of the S states, only $N+1$ states are genuinely distinguishable by asset returns. Whereas, the remaining $S-N-1$ states are asset-redundant states (i.e., asset returns in these $S-N-1$ states are identical to returns in some of the $N+1$ primary states). We accordingly partition the entire state space into $N+1$ subspaces, $\mathcal{S} = \mathcal{S}_0 \cup \ldots \cup \mathcal{S}_N$, in such a way that traded assets yield indistinguishable returns in states within each subspace $\mathcal{S}_i$. Without loss of generality, let each of N assets $\{Y_1, \ldots, Y_N\}$ load only (and indistinguishably) on states within the respective subspaces in $\{\mathcal{S}_1, \ldots, \mathcal{S}_N\}$. That is, when risks are completely disentangled, the subspace $\mathcal{S}_n$ ($n \in \{1, \ldots, N\}$) is entirely represented by any state, and is concisely defined as follows,

$$Y_{nt+1}(s) \equiv Y_{t+1}(\mathcal{S}_n) > 1 + r_H, \quad \forall s \in \mathcal{S}_n; \qquad\qquad Y_{nt+1}(s) = 0, \quad \forall s \in \mathcal{S} \backslash \mathcal{S}_n.$$

In matrix notation, the $S \times (N+1)$ payoff matrix Π of assets (in the home currency) at $t+1$ has the following expression, (columns represent payoffs of $N+1$ assets $\{B_H, Y_1, \ldots, Y_N\}$),

$$\Pi_{S\times(N+1)} = \begin{bmatrix} 1+r_{Ht} & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \ldots & \vdots & \vdots \\ 1+r_{Ht} & 0 & \ldots & 0 & 0 \\ 1+r_{Ht} & \widehat{y}_{t+1}(\mathcal{S}_1) & \ldots & 0 & 0 \\ \vdots & \vdots & \ldots & \vdots & \vdots \\ 1+r_{Ht} & \widehat{y}_{t+1}(\mathcal{S}_1) & \ldots & 0 & 0 \\ & & & & \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ & & & & \\ 1+r_{Ht} & 0 & \ldots & 0 & \widehat{y}_{t+1}(\mathcal{S}_N) \\ \vdots & \vdots & \ldots & \vdots & \vdots \\ 1+r_{Ht} & 0 & \ldots & 0 & \widehat{y}_{t+1}(\mathcal{S}_N) \end{bmatrix}. \tag{44}$$

When risks are completely disentangled as characterized by the above payoff matrix, clearly the n-th equation in (43) now involves only a single asset return $\widehat{y}_{t+1}(\mathcal{S}_n)$. System (43) now is linear in unknowns $\{\alpha_n\}$,

$$\frac{1 + \frac{\widehat{y}_{t+1}(\mathcal{S}_n)}{1+r_{Ht}}\,\alpha_n - \left(\sum_k^N \alpha_k\right)}{\widehat{y}_{t+1}(\mathcal{S}_n)\;\sum_{s\in\mathcal{S}_n} p_t(s)\widehat{m}_{Ft+1}(s)} = \frac{1+r_{Ft}}{1+r_{Ht}}, \qquad \forall n \in \{1,\ldots,N\}, \tag{45}$$

from which follows a unique solution for $\{\alpha_n\}$, and the unique associated exchange rate (42). This scenario corresponds to the complete risk disentanglement in the asset market $\mathcal{T}$ (31) of continuous settings (Section A).

Second scenario: In the second scenario, when excess returns of traded assets are sufficiently small, $\widehat{y}_{nt+1}(s) - (1 + r_{Ht}) \to 0$, $\forall s \in \mathcal{S}$, then in the limit (43) reduces to a linear system of unknowns $\{\alpha_n\}$,

$$\sum_{s\in\mathcal{S}} p_t(s)\widehat{m}_{Ft+1}(s)\left(1 - \sum_{k=1}^{N}\alpha_k\left[\frac{\widehat{y}_{kt+1}(s)}{1+r_{Ht}} - 1\right]\right)\widehat{y}_{nt+1}(s) = \frac{1+r_{Ht}}{1+r_{Ft}}, \qquad \forall n \in \{1,\ldots,N\}. \tag{46}$$

This scenario corresponds to the ubiquitous diffusion risk in continuous time settings. Therein, as the increment in time approaches zero, so do increments in returns, and the approximation underlying (46) holds as a result of Itô's lemma. By the same reason, the current scenario also corresponds to the the jump risk of insignificant magnitudes (see the discussion following Definition 1 and approximation (17)).

Entangled Risks: When not every asset risk can be singly replicated by a portfolio of traded assets, risks are entangled in asset markets. Risk entanglement then is tantamount to the genuine non-linearity of the equation system (43), resulting in multiple pricing-consistent exchange rates. Because complete risk disentanglement is mutually exclusive to risk entanglement, the latter premise prevails when both (i) asset risks are not completely spanned in asset markets, and (ii) the magnitude of risks is significant.[38] From the conceptual perspective of a discrete state and time setting, risk entanglement thus is a natural and ubiquitous premise, whereas complete risk entanglement is a special one.[39]

[38]These conditions rule out respectively the linear versions (45) and (46) of system (43). In particular, risks of significant magnitudes warrant the breakdown of the first-order (linear) Taylor approximation to the exchange rate (46), rendering the original system (43) genuinely non-linear in weights $\{\alpha_n\}$.

[39]In continuous settings (Section 2.6 and Appendix A), jump risks (of significant magnitudes) are needed to give rise to risk entanglement of Definition 1. The continuous time literature focusing on diffusion risks hence does not reveal risk entanglement effects.

Intuitively, asset risks tend to be highly entangled when there are many states yet relatively few traded assets ($S \gg N$). Quantitatively, then (43) is a system of polynomial equations of high degrees (i.e., highly non-linear) and potentially has a large number of solutions. As a result, the set of possible dynamics of consistent exchange rates is also richer and larger to be matched with the data. Therefore, a higher degree of risk entanglement offers greater a flexibility to calibrate empirical exchange rate dynamics. The same working mechanism holds for the risk entanglement in continuous settings underlying system (30).

Finally, a highly useful modeling feature of risk entanglement is its ability to create a deviation between asset return spaces denominated in home and foreign currencies, $\{B_H, Y_n\} \neq \{eB_H, eY_n\}$. This feature is exclusive to risk entanglement, and holds for both continuous and discrete settings.[40] In discrete settings, a simple example with two basis assets $\{B_H, Y(s)\}$ and entangled states ($S \equiv dim(\mathcal{S}) > 2$) illustrates this feature. As bond (constant) payoffs in its base currency are uncorrelated with any asset returns denominated in the other currency, the wedge between the return spaces $\{B_H, Y\}$ and $\{eB_H, eY\}$ essentially is captured by the correlation between the remaining country-specific risky assets,

$$Cov_t\left(\frac{Y_{t+1}}{Y_t}, \frac{e_{t+1}\,Y_{t+1}}{e_t\,Y_t}\right) = \left(\sum_{s=1}^{S} \frac{Y_{t+1}^2(s)}{Y_t^2}\frac{e_{t+1}(s)}{e_t}\right) - \left(\sum_{s=1}^{S} \frac{Y_{t+1}(s)}{Y_t}\right) \times \left(\sum_{s=1}^{S} \frac{e_{t+1}(s)\,Y_{t+1}(s)}{e_t\,Y_t}\right), \tag{47}$$

$$\text{with:} \quad \frac{e_{t+1}(s)}{e_t} = \frac{(1+r_F)}{(1-\alpha)\times(1+r_H) + \alpha \times \frac{Y_{t+1}(s)}{Y_t}}, \qquad \forall s \in \mathcal{S},$$

where the exchange rate is from the tradability constraint (42) of the foreign bond. Clearly, the ex-ante basis asset choice (i.e., the set of S payoffs $\{Y_{t+1}(s)\}$) represents the level of risk entanglement in asset markets ($S > 2$). Varying this choice regulates the above covariance, and hence, the wedge between return spaces denominated in home and foreign currencies. When a larger wedge is desirable to calibrate international price patterns, choices of the ex-ante basis asset Y can be identified in such a way that the covariance (47) vanishes, i.e., return spaces $\{B_H, Y\}$ and $\{eB_H, eY\} \equiv \{B_F, eY\}$ are orthogonal in the limit.[41] Given that two return spaces are always identical in the absence of entangled risks, this orthogonality attests to a new and striking flexibility that risk entanglement can contribute to dilute the presumed strong relationship between exchange rate and cross-country pricing dynamics. We discuss this modeling aspect of risk entanglement in

[40] In continuous settings, without risk entanglement, the two return spaces are always identical (Remarks 3 and 4).

[41] Such an orthogonality possibility is feasible only when $S > 2$ (risk entanglement). Indeed, if instead $S = 2$ (complete risk disentanglement), the basis $\{B_H, Y\}$ is complete and hence linearly spans 2-dim return vector eY. As a result, the covariance $Cov(Y, eY)$ is proportional to $Var(Y)$, and thus, does not vanish.

the next section.

In sum, risk entanglement presents a new conceptual framework to reconcile macroeconomic and exchange rate dynamics in both continuous and discrete settings.

D Projections and Exchange Rate in Diffusion Risk Settings

This appendix illustrates the relationship between two approaches to the exchange rate determination, namely (i) the portfolio representation of this paper, and (ii) the more familiar projection approach of the literature. Analytically, while we pursue the portfolio representation approach, it also draws important intuitions from the underlying projection machinery. For clarity and simplicity, we work with a diffusion risk setting in our comparative analysis.[42] The analysis also clarifies that, in continuous settings, the appropriate construction of pricing-consistent projectors necessarily concerns net growths (as opposed to gross growths) of original SDFs and asset returns.

The basic picture of the exchange rate determination is to find e_t such that all traded assets Y are priced consistently across currencies (25). Except for the simplest premise of complete markets, these pricing consistency constraints hold as equalities of conditional expectations. Therefore, the main subject of the determination, namely the exchange rate process $\frac{e_{t+dt}}{e_t}$, which positions inside expectation operators, is elusive. The projection approach tackles this challenge based on the geometric intuition that after eliminating expectation operators, these become equalities involving only the components that are in return spaces (i.e., projection). In contrast, the portfolio approach is of analytical flavor. It postulates a representation for the exchange rate growth in terms of asset return growths, enabling us to directly compute the conditional expectations (e.g., employing stochastic calculus and Itô's lemma). When asset return risk structures are complicated (e.g., entangled), the geometric projection becomes abstract, while the analytical (portfolio) approach remains explicit (as a system of equations) and is of analytical advantage.

Let country I's net (SDF) growths ($I \in \{H, F\}$) and gross asset returns (in currency H) be

[42] A full analysis concerning generic (entangled) jump-diffusion processes is involved. We refer to Maurer and Tran (2018) for technical details.

respectively,

$$
\begin{aligned}
m_{It+dt} \equiv \tfrac{M_{It+dt}-M_{It}}{M_{It}} = -r_{It}dt - \eta'_{It}dZ_t, \qquad & M_{I0}=1,\ t\in[0,\infty), \\
r_{It}\in\mathcal{F}_t,\ \eta_{It}\in\mathcal{F}_t, \qquad & I\in\{H,F\}, \\
\tfrac{Y_{t+dt}}{Y_t} = 1+\mu_{Yt}dt+\sigma'_{Yt}dZ_t, \qquad & \mu_{Yt}\in\mathcal{F}_t,\ \sigma_{Yt}\in\mathcal{F}_t,\ \ t\in[0,\infty),
\end{aligned}
\tag{48}
$$

where r_{It} and $d-$vector η_{It}, respectively are I's risk-free rate and prices of the d diffusion risks Z_t. For these specifications and for each traded asset Y, identity (30) simplifies to

$$
\sigma'_Y\left(\eta_H - \eta_F + \sigma_e\right) = 0, \qquad \forall Y\in\{Y\}. \tag{49}
$$

If markets are complete, the above identity reduces to $\sigma_e = \eta_F - \eta_H$ (because there exists a traded asset Y loading on (and only on) every risk in η_F, η_H, e_t), which then implies the well-known complete-market relationship, $\frac{e_{t+dt}}{e_t} = \frac{M_{Ht+dt}/M_{Ft+dt}}{M_{Ht}/M_{Ft}}$. Incomplete markets inspire more sophisticated techniques to evaluate this relationship.

Portfolio Representation Approach: We briefly recapitulate the essence of this approach (21). Herein, the exchange rate (and its drift, diffusion and jump components) is a function of asset returns (and their moments) (29), hence (49) become a system to determine the exchange rate. In the diffusion setting, the system is linear, yielding a unique pricing-consistent exchange rate (33) (Theorem 1). To relate this result with the projection approach, we observe that solution (33) suggests, in retrospect, an ad-hoc and explicit way to identify/reconstruct SDFs which consistently price risks across currencies given this exchange rate solution. Specifically, for every country I, we construct the projected net SDF growth processes as the unique solution of the following SDE,

$$
\frac{dM_{It\|}}{M_{It\|}} \equiv \frac{M_{It+dt\|}-M_{It\|}}{M_{It\|}} = -r_{It}dt - \eta'_{It\|}dZ_{t\|}, \quad t\in[0,\infty), \quad M_{I0\|}=1, \quad I\in\{H,F\}. \tag{50}
$$

from the prices of risk projected on asset returns, $\eta_{I\|}$. From this construction then follows a relationship (between the exchange rate and pricing dynamics) in the diffusion setting,

$$
\frac{e_{t+dt}}{e_t} = \frac{M_{H\|t+dt}/M_{F\|t+dt}}{M_{H\|t}/M_{F\|t}} \quad \text{or equivalently,} \quad \frac{de_t}{e_t} = \frac{1+\frac{dM_{H\|t}}{M_{H\|t}}}{1+\frac{dM_{F\|t}}{M_{F\|t}}} - 1.
$$

Insights from the portfolio representation approach are as follows. First, the exchange rate solution is expressed in terms of a stochastic process of exchange rate growth $\frac{de_t}{e_t}$ (33). Second, and therefore, the SDF "projectors" that prices this exchange rate solution are in fact also the stochastic processes of the growths of these projectors. Third, it is important to note the order of steps in the construction of SDF projectors: (i) the procedure is started by projecting the price-of-risk vector η_{It} onto the space of traded asset returns to obtain $\eta_{It\|}$, (ii) from this $\eta_{It\|}$ the projected net SDF growth $\frac{dM_{It\|}}{M_{It\|}}$ is uniquely constructed, and finally, (iii) the projected SDF level $M_{It\|}$ is unambiguously determined by integrating the projected net SDF growth $\frac{dM_{It\|}}{M_{It\|}}$ under standard regularity conditions.

Projection Approach: The projection approach starts from Euler pricing equations (25), $E_t[\frac{M_{Ht+dt}}{M_{Ht}}\frac{Y_{t+dt}}{Y_t}] = 1 = E_t[\frac{M_{Ft+dt}}{M_{Ft}}\frac{e_{t+dt}}{e_t}\frac{Y_{t+dt}}{Y_t}]$, for all traded asset Y. In a diffusion risk setting, as a special case of completely disentangled risks, it is always possible to form an complete asset basis $\{Y\}$, each of which loads singly on a diffusion risk $i \in Z_{\{Y\}}$ of asset returns. Therefore, the Euler pricing equations imply a risk-by-risk relationship between the exchange rate and SDFs,

$$\frac{M_{Hit+dt}}{M_{Hit}} = \frac{M_{Fit+dt}}{M_{Fit}}\frac{e_{it+dt}}{e_{it}} \implies \frac{e_{it+dt}}{e_{it}} = \frac{\frac{M_{Hit+dt}}{M_{Hit}}}{\frac{M_{Fit+dt}}{M_{Fit}}}, \quad \forall i \in Z_{\{Y\}}. \tag{51}$$

In this approach, and for diffusion settings, a SDF projector is simply a collection of original SDF components along the set of diffusion risks $Z_{\{Y\}}$ in asset returns. As a result, the prices of risks associated with this SDF projector are also collections of original ones $\{\eta_{Ii}\}$, $i \in Z_{\{Y\}}$, which indeed coincide with $\eta_{I\|}$ (50) obtained in the portfolio representation approach. This demonstrates, in diffusion settings, that the construction of SDF projectors in the portfolio representation approach is identical to the standard geometric SDF projectors (51).

It is important to note that the construction of SDF projectors in our analytical approach (50) concerns the projection of *net* SDF growths. In this regard, we elaborate on a further technical result that, in continuous settings, the distinction between SDF projectors constructed on gross and net SDF growths is relevant: SDF projectors constructed from gross SDF growths do not price asset returns denominated in respective currencies.

To see this, we first project the gross SDF growth $\widehat{m}_{It+dt} \equiv \frac{M_{It+dt}}{M_{It}}$ onto the space of gross assets returns $\{\widehat{y}_{Iit+dt}\} \equiv \left\{\frac{Y_{Iit+dt}}{Y_{Iit}}\right\}$ respectively for $I \in \{H, F\}$,

$$\widehat{m}_{H\|t+dt} = \sum_{i=1}^{N+1} \widehat{\gamma}_{Hi}\widehat{y}_{Hit+dt}, \qquad \widehat{m}_{F\|t+dt} = \sum_{i=1}^{N+1} \widehat{\gamma}_{Fi}\widehat{y}_{Fit+dt} = \frac{e_{t+dt}}{e_t}\sum_{i=1}^{N+1} \widehat{\gamma}_{Fi}\widehat{y}_{Hit+dt}. \tag{52}$$

The fact that projectors of gross SDF growths do not price asset returns in continuous settings then follows by noting that the projection (52) entails three separate matching constraints (on free term, drift term associated with dt, and volatility term associated with dZ_t). The matching of free terms (terms of 1) of both sides of the linear construction (52) implies that $\sum_{i=1}^{N+1} \widehat{\gamma}_{Ii} = 1$, or SDF projector $\widehat{m}_{I\|t+dt}$ is a proper portfolio, thus equals a gross return, of traded assets. Intuitively, it is this constraint that makes this projector unfit to price assets in continuous setting, $E_t[\widehat{m}_{I\|t+dt}\widehat{y}_{Iit+dt}] \neq 1$. This is because while projectors are co-linear with asset returns, they do not need to be a return on traded assets. Quantitatively, when we indeed enforce both the matching of free terms (i.e., $\widehat{m}_{I\|t+dt}$ necessarily is a traded gross return), and this SDF projector's pricing ability (i.e., the representation $\widehat{m}_{I\|t+dt} = 1 - r_I dt - \eta'_{I\|} dZ_t$ necessarily holds), then the construction $\widehat{m}_{I\|t+dt}$ needs to price itself,

$$E_t\left[\widehat{m}_{I\|t+dt}\widehat{m}_{I\|t+dt}\right] = 1 \implies \frac{1}{2}\eta'_{I\|}\,\eta_{I\|} = r_I.$$

Clearly, for every country I, projected prices of risks $\eta_{I\|}$ (which are collections of components of original prices of risks η_I in the diffusion risks of asset returns as seen earlier) are exogenous to I's risk-free rate r_I. Therefore, the last constraint above is spurious and is not expected to hold in general. To get around this impasse, SDF projectors (50) are constructed from net SDF growths, hence do not entail the matching of the free term (which is always absent in net growth quantities) and do price asset returns in respective currencies.

Above finding also provides a rationale for an impossibility result of Burnside and Graveline (2012) concerning SDF projectors constructed on gross SDF growths. Indeed, relationships (52) imply the impossibility result, $\frac{\widehat{m}_{H\|t+dt}}{\widehat{m}_{F\|t+dt}} \neq \frac{e_{t+dt}}{e_t}$, i.e., the ratio of SDF projectors (constructed on gross SDF growths) does not equal the exchange rate in general.[43] In retrospect, because projectors of gross SDF growths do not price asset returns in continuous settings, their ratio does not equal the exchange rate as Burnside and Graveline (2012) observe. In this paper, whenever continuous settings are concerned, we therefore work with projectors of net SDF growths.

[43] Burnside and Graveline (2012) provide an contradiction argument for this impossibility result. Assume that $\frac{\widehat{m}_{H\|t+dt}}{\widehat{m}_{F\|t+dt}} = \frac{e_{t+dt}}{e_t}$. Linear projections (52) then imply, $\frac{e_t}{e_{t+dt}} \sum_{i=1}^{k+1} \widehat{\gamma}_{Hi}\widehat{y}_{Hit+dt} = \frac{e_{t+dt}}{e_t} \sum_{i=1}^{k+1} \widehat{\gamma}_{Fi}\widehat{y}_{Hit+dt}$. Given the arbitrary (exogenous) asset returns $\{\widehat{y}_{Hit+dt}\}$, this equality is non-linear (in $\frac{e_{t+dt}}{e_t}$), and therefore, is generally violated.